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THESIS

RIEMANN'S *P*-FUNCTION

BY

CHARLES H. CHAPMAN

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# RIEMANN'S *P*-FUNCTION.

## INTRODUCTION.

The following paper was begun at the suggestion of Dr. Craig and has had the benefit of his advice and criticism. It is an attempt to set forth in a clear and readable form the properties of the *P*-function invented by Riemann and treated of in his collected works.

Part I of the present paper is devoted to a restatement in systematic order of the properties of the function as given by Riemann, with full demonstrations of certain points at which he has only hinted, and original proofs of some propositions which he considered almost self-evident, but which admit, and perhaps require, elaborate demonstrations.

Section 4 of Part I is an instance of this kind; the work therein and the table of equivalent functions are my own. The same may be said of Section 5, where, by using an expansion in powers of the variable which is known to be convergent in the proper regions, the ideas are fixed and a certain ease of statement is attained.

In proving that the *P*-function satisfies a linear differential equation of the second order I have followed Riemann, except that certain points in the Theory of Functions have been touched more lightly, as being better known to readers at the present day.

Part II is entirely my own. It is a study of the differential equation satisfied by the *P*-function from the point of view of the modern theory as originated by Fuchs and developed by his illustrious collaborators. The theorems concerning the exponents are stated and proved as properties of the indicial equation in Section 4. Section 3 is an elaborate study of the transformation  $x' = \frac{ex + f}{gx + h}$ ; Section 5 is devoted to obtaining the coefficients of the differential equation; and in subsequent sections the Spherical Harmonics, Toroidal functions and Bessel's functions are expressed as *P*-functions, while, in conclusion, the *P*-function itself is expressed as a Hypergeometric Series.

## PART I.

SECTION 1.—*Definition of the P-function.*

Conceive that a function exists which has the following properties :

1. It is finite and continuous throughout the plane of imaginary quantities, except at the points  $x = a$ ,  $x = b$ ,  $x = c$ .
2. Between any three branches of the function,  $P$ ,  $P'$ ,  $P''$ , there exists a linear relation with constant coefficients,

$$c'P + c''P' + c'''P'' = 0.$$

3. The function may be put in any of the forms

$$C_a P^{(\alpha)} + C_{a'} P^{(\alpha')}, \quad C_b P^{(\beta)} + C_{b'} P^{(\beta')}, \quad C_c P^{(\gamma)} + C_{c'} P^{(\gamma')},$$

where  $C_a$ ,  $C_{a'}$ ,  $C_b$ ,  $C_{b'}$ ,  $C_c$ ,  $C_{c'}$  are constants, and the expressions  $P^{(\alpha)}(x-a)^{-\alpha}$ ,  $P^{(\alpha')}(x-a)^{-\alpha'}$  become neither zero nor infinite when  $x=a$ ; likewise  $P^{(\beta)}(x-b)^{-\beta}$ ,  $P^{(\beta')}(x-b)^{-\beta'}$  are neither zero nor infinite for  $x=b$ , and  $P^{(\gamma)}(x-c)^{-\gamma}$ ,  $P^{(\gamma')}(x-c)^{-\gamma'}$  are neither zero nor infinite for  $x=c$ .

By these properties the *P*-function is completely defined, except that it contains two arbitrary constants. It is designated by the symbol

$$P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} x \right\}.$$

SECTION 2.—*The quantities  $\alpha$ ,  $\alpha'$ ,  $\beta$ ,  $\beta'$ ,  $\gamma$ ,  $\gamma'$ .*

These quantities may be anything whatever, subject to the conditions—

1. None of the differences  $\alpha - \alpha'$ ,  $\beta - \beta'$ ,  $\gamma - \gamma'$  shall be an integer.
2. The sum of all the quantities is constantly unity, i. e.,

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

SECTION 3.—*Properties of the P-function.*

1. The first three vertical columns may be interchanged at pleasure. For, when the defining conditions are applied to the three functions so obtained, no distinction can be observed between them; hence they are identical, provided the conditions actually define a function.

2. In the same way we see that  $\alpha$  may be interchanged with  $\alpha'$ ,  $\beta$  with  $\beta'$ , and  $\gamma$  with  $\gamma'$ .

3. Let  $x$  be replaced by  $x'$ , a rational linear function of  $x$ , so taken that when

$$\begin{aligned} x=a, \quad x' &= a', \\ x=b, \quad x' &= b', \\ x=c, \quad x' &= c', \end{aligned}$$

then the two functions  $P \left\{ \begin{smallmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right\}$  and  $P \left\{ \begin{smallmatrix} a' & b' & c' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x' \right\}$  are equal.

By this transformation, to be fully developed later on, every  $P$ -function may be expressed in terms of another, whose singular points are 0,  $\infty$ , 1. But every function having the same  $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$  may thus be put in the form

$P \left\{ \begin{smallmatrix} 0 & \infty & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right\}$ , and our definition will then make no distinction between

them; that is, all  $P$ -functions having the same exponents,  $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ ,

may be reduced to the same function  $P \left\{ \begin{smallmatrix} 0 & \infty & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right\}$ , which may be briefly

written  $P \left\{ \begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right\}$ .

According to the linear expression in  $x$  which we choose for the variable, the points 0,  $\infty$ , 1 may appear in six different ways, corresponding to six modes of propagation of the function in the plane of  $x$ . They are

$$\begin{aligned} P \left\{ \begin{smallmatrix} 0 & \infty & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right\}; & \quad P \left\{ \begin{smallmatrix} \infty & 0 & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \frac{1}{x} \right\}; & \quad P \left\{ \begin{smallmatrix} 1 & \infty & 0 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} 1-x \right\}; \\ P \left\{ \begin{smallmatrix} 0 & 1 & \infty \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \frac{x}{x-1} \right\}; & \quad P \left\{ \begin{smallmatrix} \infty & 1 & 0 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \frac{x-1}{x} \right\}; & \quad P \left\{ \begin{smallmatrix} 1 & 0 & \infty \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \frac{1}{1-x} \right\}. \end{aligned}$$

#### SECTION 4.—Transformation of Exponents.

We shall have, by definition, the product  $P^{(\alpha+\delta)}(x-a)^{-\alpha-\delta}$  neither 0 nor  $\infty$  for  $x=a$ : hence, consistently with all that precedes, we may write, denoting by  $P_1$  a new  $P$ -function,



$$\begin{aligned}
P_1^{(\alpha+\delta)} &= (x-a)^\delta P^{(\alpha)}(x-b)^{-\delta}, \text{ if we choose; and} \\
P_1^{(\alpha'+\delta)} &= (x-a)^\delta P^{(\alpha')}(x-b)^{-\delta}; \quad P_1^{(\beta-\delta)} = (x-b)^{-\delta} P^{(\beta)}(x-a)^\delta; \\
P_1^{(\beta'-\delta)} &= (x-b)^{-\delta} P^{(\beta')}(x-a)^\delta; \quad P_1^{(\gamma)} = (x-a)^\delta (x-b)^{-\delta} P^{(\gamma)}; \\
P_1^{(\gamma')} &= (x-a)^\delta (x-b)^{-\delta} P^{(\gamma')}.
\end{aligned}$$

Observing that the left-hand members of these equations are the constituent

branches of the function  $P_1 \left\{ \begin{smallmatrix} a & b & c \\ \alpha + \delta & \beta - \delta & \gamma & x \\ \alpha' + \delta & \beta' - \delta & \gamma' \end{smallmatrix} \right\}$ , we have the relation

$$P_1 \left\{ \begin{smallmatrix} a & b & c \\ \alpha + \delta & \beta - \delta & \gamma & x \\ \alpha' + \delta & \beta' - \delta & \gamma' \end{smallmatrix} \right\} = \left( \frac{x-a}{x-b} \right)^\delta P \left\{ \begin{smallmatrix} a & b & c \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \right\}.$$

If the  $P$ -function be in the reduced form

$$P \left\{ \begin{smallmatrix} 0 & \infty & 1 \\ \alpha & \beta & \gamma & x' \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \right\},$$

then in the region of the point  $\infty$  either branch has the form

$$\left( \frac{1}{x'} \right)^\delta \left( a_0 + \frac{a_1}{x'} + \frac{a_2}{x'^2} + \dots \right) = \left( \frac{1}{x'} \right)^\delta Y^\delta;$$

because  $P \left\{ \begin{smallmatrix} a & b & c \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \right\}$  had in the region of the point  $b$  the form

$$(x-b)^\delta [a'_0 + a'_1(x-b) + a'_2(x-b)^2 + \dots],$$

and a transformation of such a nature that when  $x=b$ ,  $x'$  shall become infinite, can only be of the form  $x-b = \frac{1}{x'}$ ; hence the transformed function in the region of the point  $\infty$  has the form

$$\left( \frac{1}{x} \right)^\delta \left[ a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots \right] = \left( \frac{1}{x} \right)^\delta Y^{(\delta)}.$$

It follows that

$$P \left\{ \begin{smallmatrix} \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' \end{smallmatrix} \right\} (1-x)^{\alpha} x^{\beta}$$

will, in the region of the point  $\infty$ , be of the form

$$c_\beta \frac{(1-x)^{\alpha} x^{\beta}}{x^{\beta}} Y^{(\beta)} + c_{\beta'} \frac{(1-x)^{\alpha} x^{\beta}}{x^{\beta'}} Y^{(\beta')},$$

where  $Y^{(\beta)}$  and  $Y^{(\beta')}$  are neither zero nor infinite for  $x = \infty$ . Clearly then, putting this in the form  $c_\beta P^{(\beta)} + c_{\beta'} P^{(\beta')}$ , we see that  $P^{(\beta)} x^{\beta-\delta-\epsilon}$  and  $P^{(\beta')} x^{\beta'-\delta-\epsilon}$  are neither 0 nor  $\infty$  at the point  $\infty$ ; hence

$$P \left\{ \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \middle| x \right\} (1-x)^{\delta} x^{\epsilon} = P \left\{ \begin{matrix} \alpha + \delta & \beta - \delta - \epsilon & \gamma + \epsilon \\ \alpha' + \delta & \beta' - \delta - \epsilon & \gamma' + \epsilon \end{matrix} \middle| x \right\};$$

the first and last exponents being transformed by the rule found above.

Here  $\delta$  and  $\epsilon$  may have any values whatever, and this remark permits us to draw the following inference:

The values of any two of the exponents may be changed at pleasure by introducing proper multipliers; but the sum  $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma'$  must remain unchanged, and always equal to 1. The differences  $\alpha - \alpha'$ ,  $\beta - \beta'$ ,  $\gamma - \gamma'$  must also remain unaltered in absolute magnitude. In other words, the product of a  $P$ -function by factors which fulfil the above conditions may be expressed as a  $P$ -function.

Again,  $P$ -functions, in which the differences  $\alpha - \alpha'$ ,  $\beta - \beta'$ ,  $\gamma - \gamma'$  are the same, can differ only by determinate factors, as the following table will more fully illustrate. The transformations involved will be considered in Part II.

$$P \left( \begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \middle| x \right) = \begin{bmatrix} (1-x)^{\gamma} x^{\alpha} P \left( \begin{matrix} 0 & \beta + \alpha + \gamma & 0 \\ \alpha' - \alpha & \beta' + \alpha + \gamma & \gamma' - \gamma \end{matrix} \middle| x \right), \\ (1-x)^{\gamma} x^{\alpha'} P \left( \begin{matrix} \alpha - \alpha' & \beta + \alpha' + \gamma' & \gamma - \gamma' \\ 0 & \beta' + \alpha' + \gamma' & 0 \end{matrix} \middle| x \right), \\ (1-x)^{\gamma} x^{\alpha'} P \left( \begin{matrix} \alpha - \alpha' & \beta + \alpha' + \gamma & 0 \\ 0 & \beta' + \alpha' + \gamma & \gamma' - \gamma \end{matrix} \middle| x \right), \\ (1-x)^{\gamma} x^{\alpha} P \left( \begin{matrix} 0 & \beta + \alpha + \gamma' & \gamma - \gamma' \\ \alpha' - \alpha & \beta' + \alpha + \gamma' & 0 \end{matrix} \middle| x \right). \end{bmatrix}$$

$$P \left( \begin{matrix} \beta & \alpha & \gamma \\ \beta' & \alpha' & \gamma' \end{matrix} \middle| \frac{1}{x} \right) = \begin{bmatrix} \left( \frac{x-1}{x} \right)^{\gamma} \frac{1}{x^{\beta}} P \left( \begin{matrix} 0 & \alpha + \beta + \gamma & 0 \\ \beta' - \beta & \alpha' + \beta + \gamma & \gamma' - \gamma \end{matrix} \middle| \frac{1}{x} \right), \\ \left( 1 - \frac{1}{x} \right)^{\gamma} \frac{1}{x^{\beta'}} P \left( \begin{matrix} \beta - \beta' & \alpha + \beta' + \gamma' & \gamma - \gamma' \\ 0 & \alpha' + \beta' + \gamma' & 0 \end{matrix} \middle| \frac{1}{x} \right), \\ \left( 1 - \frac{1}{x} \right)^{\gamma} \frac{1}{x^{\beta'}} P \left( \begin{matrix} \beta - \beta' & \alpha + \beta' + \gamma & 0 \\ 0 & \alpha' + \beta' + \gamma & \gamma' - \gamma \end{matrix} \middle| \frac{1}{x} \right), \\ \left( 1 - \frac{1}{x} \right)^{\gamma} \frac{1}{x^{\beta}} P \left( \begin{matrix} 0 & \alpha + \beta + \gamma' & \gamma - \gamma' \\ \beta' - \beta & \alpha' + \beta + \gamma' & 0 \end{matrix} \middle| \frac{1}{x} \right), \end{bmatrix}$$

$$P\left(\begin{matrix} \alpha & \gamma & \beta \\ \alpha' & \gamma' & \beta' \end{matrix} \middle| \frac{x}{x-1}\right) = \begin{bmatrix} \left(\frac{1}{1-x}\right)^{\beta} \left(\frac{x}{x-1}\right)^{\alpha} P\left(\begin{matrix} 0 & \gamma + \alpha + \beta & 0 \\ \alpha' - \alpha & \gamma' + \alpha + \beta & \beta' - \beta \end{matrix} \middle| \frac{x}{x-1}\right), \\ \left(\frac{1}{1-x}\right)^{\beta'} \left(\frac{x}{x-1}\right)^{\alpha'} P\left(\begin{matrix} \alpha - \alpha' & \gamma + \alpha' + \beta' & \beta - \beta' \\ 0 & \gamma' + \alpha' + \beta' & 0 \end{matrix} \middle| \frac{x}{x-1}\right), \\ \left(\frac{1}{1-x}\right)^{\beta} \left(\frac{x}{x-1}\right)^{\alpha'} P\left(\begin{matrix} \alpha - \alpha' & \gamma + \alpha' + \beta & 0 \\ 0 & \gamma' + \alpha' + \beta & \beta' - \beta \end{matrix} \middle| \frac{x}{x-1}\right), \\ \left(\frac{1}{1-x}\right)^{\beta'} \left(\frac{x}{x-1}\right)^{\alpha} P\left(\begin{matrix} 0 & \gamma + \alpha + \beta' & \beta - \beta' \\ \alpha' - \alpha & \gamma' + \alpha + \beta' & 0 \end{matrix} \middle| \frac{x}{x-1}\right). \end{bmatrix}$$

$$P\left(\begin{matrix} \gamma & \alpha & \beta \\ \gamma' & \alpha' & \beta' \end{matrix} \middle| \frac{x-1}{x}\right) = \begin{bmatrix} \left(\frac{1}{x}\right)^{\beta} \left(\frac{x-1}{x}\right)^{\gamma} P\left(\begin{matrix} 0 & \alpha + \beta + \gamma & 0 \\ \gamma' - \gamma & \alpha' + \beta + \gamma & \beta' - \beta \end{matrix} \middle| \frac{x-1}{x}\right), \\ \left(\frac{1}{x}\right)^{\beta'} \left(\frac{x-1}{x}\right)^{\gamma'} P\left(\begin{matrix} \gamma - \gamma' & \alpha + \beta' + \gamma' & \beta - \beta' \\ 0 & \alpha' + \beta' + \gamma' & 0 \end{matrix} \middle| \frac{x-1}{x}\right), \\ \left(\frac{1}{x}\right)^{\beta'} \left(\frac{x-1}{x}\right)^{\gamma} P\left(\begin{matrix} 0 & \alpha + \beta' + \gamma & \beta - \beta' \\ \gamma' - \gamma & \alpha' + \beta' + \gamma & 0 \end{matrix} \middle| \frac{x-1}{x}\right), \\ \left(\frac{1}{x}\right)^{\beta} \left(\frac{x-1}{x}\right)^{\gamma'} P\left(\begin{matrix} \gamma - \gamma' & \alpha + \beta + \gamma' & 0 \\ 0 & \alpha' + \beta + \gamma' & \beta' - \beta \end{matrix} \middle| \frac{x-1}{x}\right). \end{bmatrix}$$

$$P\left(\begin{matrix} \gamma & \beta & \alpha \\ \gamma' & \beta' & \alpha' \end{matrix} \middle| 1-x\right) = \begin{bmatrix} x^{\alpha} (1-x)^{\gamma} P\left(\begin{matrix} 0 & \beta + \alpha + \gamma & 0 \\ \gamma' - \gamma & \beta' + \alpha + \gamma & \alpha' - \alpha \end{matrix} \middle| 1-x\right), \\ x^{\alpha'} (1-x)^{\gamma} P\left(\begin{matrix} \gamma - \gamma' & \beta + \alpha' + \gamma' & \alpha - \alpha' \\ 0 & \beta' + \alpha' + \gamma' & 0 \end{matrix} \middle| 1-x\right), \\ x^{\alpha'} (1-x)^{\gamma} P\left(\begin{matrix} 0 & \beta + \alpha' + \gamma & \alpha - \alpha' \\ \gamma' - \gamma & \beta' + \alpha' + \gamma & 0 \end{matrix} \middle| 1-x\right), \\ x^{\alpha} (1-x)^{\gamma'} P\left(\begin{matrix} \gamma - \gamma' & \beta + \alpha + \gamma' & 0 \\ 0 & \beta' + \alpha + \gamma' & \alpha' - \alpha \end{matrix} \middle| 1-x\right). \end{bmatrix}$$

$$P\left(\begin{matrix} \beta & \gamma & \alpha \\ \beta' & \gamma' & \alpha' \end{matrix} \middle| \frac{1}{1-x}\right) = \begin{bmatrix} \left(\frac{x}{x-1}\right)^{\alpha} \left(\frac{1}{1-x}\right)^{\beta} P\left(\begin{matrix} 0 & \gamma + \alpha + \beta & 0 \\ \beta' - \beta & \gamma' + \alpha + \beta & \alpha' - \alpha \end{matrix} \middle| \frac{1}{1-x}\right), \\ \left(\frac{x}{x-1}\right)^{\alpha'} \left(\frac{1}{1-x}\right)^{\beta'} P\left(\begin{matrix} \beta - \beta' & \gamma + \alpha' + \beta' & \alpha - \alpha' \\ 0 & \gamma' + \alpha' + \beta' & 0 \end{matrix} \middle| \frac{1}{1-x}\right), \\ \left(\frac{x}{x-1}\right)^{\alpha'} \left(\frac{1}{1-x}\right)^{\beta} P\left(\begin{matrix} 0 & \gamma + \alpha' + \beta & \alpha - \alpha' \\ \beta' - \beta & \gamma' + \alpha' + \beta & 0 \end{matrix} \middle| \frac{1}{1-x}\right), \\ \left(\frac{x}{x-1}\right)^{\alpha} \left(\frac{1}{1-x}\right)^{\beta'} P\left(\begin{matrix} \beta - \beta' & \gamma + \alpha + \beta' & 0 \\ 0 & \gamma' + \alpha + \beta' & \alpha' - \alpha \end{matrix} \middle| \frac{1}{1-x}\right). \end{bmatrix}$$

SECTION 5.—Case where  $\alpha' - \alpha = \frac{1}{2}$ ;  $P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \beta & \gamma & x \\ \frac{1}{2} & \beta' & \gamma' \end{matrix} \right\}$ .

By Section 4 this may always be reduced to the particular case

$$\alpha = 0, \alpha' = \frac{1}{2}.$$

We shall then have

$$\beta + \beta' + \gamma + \gamma' = \frac{1}{2},$$

and the function is

$$P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \beta & \gamma & x \\ \frac{1}{2} & \beta' & \gamma' \end{matrix} \right\}.$$

In the region of the point  $x = 0$  the function may by definition be put in the form

$$c_1(a_0 + a_1x + a_2x^2 + \dots) + c'_1x^{\frac{1}{2}}(a'_0 + a'_1x + a'_2x^2 + \dots); \quad (1)$$

in the region of the point infinity,

$$c_2\left(\frac{1}{x}\right)^{\beta}\left(b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \dots\right) + c'_2\left(\frac{1}{x}\right)^{\beta'}\left(b'_0 + \frac{b'_1}{x} + \frac{b'_2}{x^2} + \dots\right), \quad (2)$$

and in the region of the point  $x = 1$ ,

$$c_3(x-1)^{\gamma}[d_0 + d_1(x-1) + d_2(x-1)^2 + \dots] + c'_3(x-1)^{\gamma'}[d'_0 + d'_1(x-1) + d'_2(x-1)^2 + \dots], \quad (3)$$

the series being convergent.

If now we write

$$x^{\frac{1}{2}} = z, \text{ or } x = z^2,$$

the series (1) will no longer have the point  $z = 0$  for a branch point, since it will contain no fractional exponents.

The series (2) will become

$$c_2\left(\frac{1}{z}\right)^{2\beta}\left[b_0 + \frac{b_1}{z^2} + \frac{b_2}{z^4} + \dots\right] + c'_2\left(\frac{1}{z}\right)^{2\beta'}\left[b'_0 + \frac{b'_1}{z^2} + \dots\right],$$

and the transformed *P*-function will have the point  $\infty$  for a branch point with the exponents  $2\beta, 2\beta'$ .

The series (3) becomes

$$c_3(z-1)^{\gamma}(z+1)^{\gamma}[d_0 + d_1(z^2-1) + d_2(z^2-1)^2 + \dots] + c'_3(z-1)^{\gamma'}(z+1)^{\gamma'}[d'_0 + d'_1(z^2-1) + \dots];$$

for which the points  $+1$  and  $-1$  are now branch points, each with the exponents  $\gamma, \gamma'$ .

Hence the substitution  $x^\dagger = z$  yields a *P*-function whose branch points are  $+1, -1$  and  $\infty$ , with the exponents  $\gamma, \gamma'; \gamma, \gamma'; 2\beta, 2\beta'$  respectively; and inasmuch as we have altered only the mode of propagation of the function, we may write

$$P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \beta & \gamma \\ \frac{1}{2} & \beta' & \gamma' \end{matrix} \middle| x \right\} = P \left\{ \begin{matrix} 1 & \infty & -1 \\ \gamma & 2\beta & \gamma \\ \gamma' & 2\beta' & \gamma' \end{matrix} \middle| \sqrt{x} \right\}.$$

The sum of the exponents is now

$$2(\gamma + \gamma') + 2(\beta + \beta') = 1.$$

If the difference  $2\beta - 2\beta'$  is an integer, the function

$$P \left\{ \begin{matrix} 1 & \infty & -1 \\ \gamma & 2\beta & \gamma \\ \gamma' & 2\beta' & \gamma' \end{matrix} \middle| \sqrt{x} \right\}$$

no longer corresponds to the definition of a *P*-function.

$$\text{SECTION 6. } -\alpha' - \alpha = \frac{1}{2}; \beta' - \beta = \frac{1}{2}; P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & 0 & \gamma \\ \frac{1}{2} & \frac{1}{2} & \gamma' \end{matrix} \middle| x \right\}.$$

In the region of the point 0 the form of the function is

$$c_1 [a_0 + a_1 x + \dots] + c'_1 x^\dagger [a'_0 + a'_1 x + \dots]. \quad (1)$$

In the region of the point  $\infty$ ,

$$c_2 \left[ b_0 + \frac{b_1}{x} + \dots \right] + c'_2 \left( \frac{1}{x} \right)^\dagger \left[ b'_0 + \frac{b'_1}{x} + \dots \right]; \quad (2)$$

and in the region of the point 1,

$$c_3 (x-1)^\gamma [d_0 + d_1 (x-1) + \dots] + c'_3 (x-1)^{\gamma'} [d'_0 + d'_1 (x-1) + \dots]. \quad (3)$$

If in these we make  $x^\dagger = z$ , then the points  $x=0$  and  $x=\infty$  are no longer branch points for (1) and (2) respectively, while (3) becomes

$$c_3(x-\rho)^\gamma(x-\rho^2)^\gamma(x-\rho^3)^\gamma[d_0+d_1(x^3-1)+\dots] \\ + c'_3(x-\rho)^\gamma(x-\rho^2)^\gamma(x-\rho^3)^\gamma[d'_0+d_1(x^3-1)+\dots];$$

$\rho$  being a cube root of unity. This has for branch points  $\rho, \rho^2, \rho^3$  each with the exponents  $\gamma, \gamma'$ .

Hence by the transformation  $x^3 = z$  we obtain

$$P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & 0 & \gamma & x \\ \frac{1}{3} & \frac{1}{3} & \gamma' \end{matrix} \right\} = P \left\{ \begin{matrix} \rho & \rho^2 & \rho^3 \\ \gamma & \gamma & \gamma & x^3 \\ \gamma' & \gamma' & \gamma' \end{matrix} \right\}.$$

Here we must have  $\gamma + \gamma' = \frac{1}{3}$ , and therefore  $3(\gamma + \gamma') = 1$ , as it should.

#### SECTION 7.—Applications of the Preceding Transformations.

Let the differences  $\alpha - \alpha', \beta - \beta', \gamma - \gamma'$  be denoted respectively by  $\lambda, \mu, \nu$ , and the function  $P \left( \begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right)$  by  $P(\lambda, \mu, \nu, x)$ .

Since the values 0,  $\infty$ , 1 for  $1-x$  correspond to the values 1,  $\infty$ , 0 for  $x$ , therefore, by Section 3,

$$P(\mu, \nu, \frac{1}{2}, 1-x) = P(\frac{1}{2}, \nu, \mu, x) = P \left\{ \begin{matrix} -1 & \infty & 1 \\ \beta & 2\gamma & \beta & \sqrt{x} \\ \beta' & 2\gamma' & \beta' \end{matrix} \right\}$$

by Section 5. Noting further that the values  $-1, \infty, 1$  for  $\sqrt{x}$  correspond to 0,  $\infty, 1$  for  $\frac{\sqrt{x+1}}{2}$ , we obtain finally

$$P(\mu, \nu, \frac{1}{2}, 1-x) = P(\frac{1}{2}, \nu, \mu, x) = P\left(\mu, 2\nu, \mu, \frac{\sqrt{x+1}}{2}\right). \quad (1)$$

By these relations, *P*-functions which have two differences the same, or one difference equal to  $\frac{1}{2}$ , are mutually expressible.

Again, observing that the values 0,  $\infty$ , 1 for  $x$  correspond to 1, 0,  $\infty$  for  $\frac{1}{1-x}$ , it follows that

$$P\left(\frac{1}{2}, \frac{1}{2}, \nu, x\right) = P\left(\frac{1}{2}, \nu, \frac{1}{2}, \frac{1}{1-x}\right) = P \left\{ \begin{matrix} 1 & \rho & \rho^2 \\ \gamma & \gamma & \gamma & x^3 \\ \gamma' & \gamma' & \gamma' \end{matrix} \right\}$$



if  $\nu = \gamma - \gamma'$ . And since the values  $1, \rho, \rho^3$  of  $x^{\frac{1}{3}}$  correspond to  $0, \infty, 1$  for  $\frac{\sqrt[3]{x}-1}{\rho^3-\rho\sqrt[3]{x}}$ , we find that

$$P \left\{ \begin{matrix} 1 & \rho & \rho^3 \\ \gamma & \gamma & \gamma \\ \gamma' & \gamma' & \gamma' \end{matrix} \middle| x^{\frac{1}{3}} \right\} = P \left( \nu, \nu, \nu, \frac{x^{\frac{1}{3}}-1}{\rho^3-\rho x^{\frac{1}{3}}} \right).$$

Put  $\frac{x^{\frac{1}{3}}-1}{\rho^3-\rho x^{\frac{1}{3}}} = x_1$ ; then, because two differences are the same in  $P(\nu, \nu, \nu, x_1)$ , we may apply equation (1) and thus obtain

$$P(\nu, \nu, \nu, x) = P\left(\frac{1}{2}, \frac{\nu}{2}, \nu, (2x_1-1)^2\right) = P\left(\frac{1}{2}, \nu, \frac{\nu}{2}, \frac{x_2}{x_2-1}\right),$$

if  $(2x_1-1)^2 = x_2$ .

And again

$$\begin{aligned} P\left(\frac{1}{2}, \nu, \frac{\nu}{2}, \frac{x_2}{x_2-1}\right) &= P \left\{ \begin{matrix} -1 & \infty & 1 \\ \frac{\gamma}{2} & 2\gamma & \frac{\gamma}{2} \\ \frac{\gamma'}{2} & 2\gamma' & \frac{\gamma'}{2} \end{matrix} \middle| \sqrt{\frac{x_2}{x_2-1}} \right\} \\ &= P\left(\frac{\nu}{2}, 2\nu, \frac{\nu}{2}, \frac{x_2+1}{2}\right); \text{ if } \sqrt{\frac{x_2}{x_2-1}} = x_3. \end{aligned}$$

Writing  $\frac{1}{1-x} = x_4$ , we find that

$$P\left(\frac{1}{2}, \nu, \frac{1}{2}, x_4\right) = P\left(\frac{1}{2}, \frac{\nu}{2}, \frac{1}{2}, (2x_4-1)^2\right)$$

by (1); and if

$$\begin{aligned} (2x_4-1)^2 = x_5, \quad P\left(\frac{1}{2}, \frac{\nu}{2}, \frac{1}{2}, x_5\right) &= P\left(\frac{1}{2}, \frac{1}{2}, \frac{\nu}{2}, \frac{x_5}{x_5-1}\right) \\ &= P\left(\frac{\nu}{2}, \frac{3}{2}, \frac{\nu}{2}, \frac{\sqrt{x_5}+1}{2}\right), \text{ if } \frac{x_2}{x_5-1} = x_4. \end{aligned}$$

Thus we have

$$\begin{aligned} &P(\nu, \nu, \nu, x_1), \quad P\left(\frac{1}{2}, \frac{\nu}{2}, \nu, x_2\right), \quad P\left(\frac{\nu}{2}, 2\nu, \frac{\nu}{2}, \frac{x_3+1}{2}\right), \\ &P\left(\frac{1}{2}, \nu, \frac{1}{2}, x_4\right); \quad P\left(\frac{1}{2}, \frac{\nu}{2}, \frac{1}{2}, x_5\right), \quad P\left(\frac{\nu}{2}, \frac{3}{2}, \frac{\nu}{2}, \frac{\sqrt{x_5}+1}{2}\right), \end{aligned} \quad (2)$$

all mutually expressible.

Making  $\mu = \nu$  in (1) we find also

$$P\left(\frac{1}{2}, \nu, \nu, x\right) = P\left(\nu, 2\nu, \nu, \frac{\sqrt{x}+1}{2}\right), \quad (3)$$

and making  $\mu = \frac{1}{2}$ ,

$$P\left(\frac{1}{2}, \nu, \frac{1}{2}, x\right) = P\left(\frac{1}{2}, 2\nu, \frac{1}{2}, \frac{\sqrt{x}+1}{2}\right). \quad (4)$$

The function  $P\left(\frac{1}{2}, \frac{1}{2}, \nu, x\right) = P\left(\nu, 1, \nu, \frac{\sqrt{x}+1}{2}\right)$  by (1), and writing  $\frac{\sqrt{x}+1}{2} = x$ , this becomes  $P(\nu, 1, \nu, x)$ , and by making  $x = \frac{x_1}{x_1-1}$ , it again transforms to  $P(\nu, \nu, 1, x)$ .

(We shall recur to this function in Part II.)

#### SECTION 8.—Study of the Reduced Function.

To study more completely the reduced function, let a cut be drawn from 1 to  $+\infty$  and from  $-\infty$  to 0; so long as the cut is not crossed,  $P\left(\begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x\right)$  is uniform and continuous, and  $P^{(\alpha)}, P^{(\alpha')}, P^{(\beta)}, P^{(\beta')}, P^{(\gamma)}, P^{(\gamma')}$  are uniform functions of  $x$  whose values are known, except constant factors depending on  $c_\alpha, c_{\alpha'}, c_\beta, c_{\beta'}, c_\gamma, c_{\gamma'}$ , when  $P$  is known. By hypothesis

$$\begin{aligned} P &= c_\alpha P^\alpha + c_{\alpha'} P^{\alpha'} \\ &= c_\alpha x_\alpha Y^\alpha + c_{\alpha'} x_{\alpha'} Y^{\alpha'}, \end{aligned}$$

where  $Y^\alpha, Y^{\alpha'}$  are uniform in the region of the point 0. Hence if  $x$  makes a tour around the point 0, we shall have

$$P' = c_\alpha e^{2\pi i} P^\alpha + c_{\alpha'} e^{2\pi i} P^{\alpha'}.$$

The determinant

$$\begin{vmatrix} c_\alpha & c_{\alpha'} \\ c_\alpha e^{2\pi i} & c_{\alpha'} e^{2\pi i} \end{vmatrix}$$

is not zero, since  $\alpha' = \alpha$  is not an integer, and therefore  $P^\alpha$  and  $P^{\alpha'}$  can be expressed linearly in terms of  $P, P'$ . The same is true of  $P^\beta, P^{\beta'}$  and  $P^\gamma, P^{\gamma'}$ . It follows that  $P^\alpha, P^{\alpha'}$  can be expressed linearly with constant coefficients in terms of  $P^\beta, P^{\beta'}$  or  $P^\gamma, P^{\gamma'}$ . Accordingly we make the assumptions

$$\begin{aligned} P^\alpha &= \alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'} = \alpha_\gamma P^\gamma + \alpha_{\gamma'} P^{\gamma'}, \\ P^{\alpha'} &= \alpha'_\beta P^\beta + \alpha'_{\beta'} P^{\beta'} = \alpha'_\gamma P^\gamma + \alpha'_{\gamma'} P^{\gamma'}. \end{aligned}$$

We may express these equations symbolically thus:  $\begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix} = T \begin{Bmatrix} P^\beta \\ P^{\beta'} \end{Bmatrix}$  where  $T$

denotes the substitution  $\begin{Bmatrix} \alpha_\beta & \alpha_{\beta'} \\ \alpha'_\beta & \alpha'_{\beta'} \end{Bmatrix}$ , and from this

$$T^{-1} \begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix} = \begin{Bmatrix} P^\beta \\ P^{\beta'} \end{Bmatrix}.$$

If, now,  $x$  makes a tour around the point 0, we shall have, denoting by  $B$  the effect on  $\begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix}$ ,

$$\begin{aligned} B \begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix} &= \begin{Bmatrix} e^{2\pi\beta t} & 0 \\ 0 & e^{2\pi\beta' t} \end{Bmatrix} T \begin{Bmatrix} P^\beta \\ P^{\beta'} \end{Bmatrix} = T \begin{Bmatrix} e^{2\pi\beta t} & 0 \\ 0 & e^{2\pi\beta' t} \end{Bmatrix} \begin{Bmatrix} P^\beta \\ P^{\beta'} \end{Bmatrix} \\ &= T \begin{Bmatrix} e^{2\pi\beta t} & 0 \\ 0 & e^{2\pi\beta' t} \end{Bmatrix} T^{-1} \begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix}. \end{aligned}$$

Since  $T$  is evidently commutative with  $\begin{Bmatrix} e^{2\pi\beta t} & 0 \\ 0 & e^{2\pi\beta' t} \end{Bmatrix}$ ; hence, separating symbols, we have for  $\begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix}$

$$B = T \begin{Bmatrix} e^{2\pi\beta t} & 0 \\ 0 & e^{2\pi\beta' t} \end{Bmatrix} T^{-1}.$$

Likewise if  $C$  denotes the effect upon  $\begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix}$  of a circuit around the point 1, we have

$$C = T_1 \begin{Bmatrix} e^{2\pi\gamma t} & 0 \\ 0 & e^{2\pi\gamma' t} \end{Bmatrix} T_1^{-1},$$

where  $T_1 = \begin{Bmatrix} \alpha_\gamma & \alpha_{\gamma'} \\ \alpha'_\gamma & \alpha'_{\gamma'} \end{Bmatrix}$ .

Finally, if  $A$  denote the substitution caused in  $\begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix}$  by a tour around the point 0, we know that

$$CBA \begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix} = \begin{Bmatrix} P^\alpha \\ P^{\alpha'} \end{Bmatrix},$$

since there are no other branch points. It follows that

$$1 = e^{2\pi(\alpha + \alpha' + \beta + \beta' + \gamma + \gamma')}$$

by forming the products of the determinants of the substitutions. This is consistent with the hypothesis that

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1.$$

We may now further investigate the quantities

$$\alpha_\beta, \alpha_{\beta'}, \alpha'_\beta, \alpha'_{\beta'}, \alpha_\gamma, \alpha_{\gamma'}, \alpha'_\gamma, \alpha'_{\gamma'}$$

as follows: A negative tour around the point 0 changes  $P^\alpha$  to  $e^{-2\pi\alpha t} P^\alpha$ , but  $P^\alpha = \alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'}$ , hence this tour changes  $\alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'}$  to  $e^{-2\pi\alpha t} (\alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'})$ .

Making now a negative tour around the point  $\infty$ , and denoting its effect on  $P^a$  by  $S$ , we have

$$e^{-2\pi ai}SP^a = e^{-2\pi ai}(\alpha_\beta e^{-2\pi\beta i}P^\beta + \alpha_{\beta'} e^{-2\pi\beta' i}P^{\beta'}).$$

But the combined effect of the two tours is that of a positive tour around the point 1; therefore, since  $P^a = \alpha_\gamma P^\gamma + \alpha_{\gamma'} P^{\gamma'}$ , we have

$$e^{-2\pi ai}SP^a = \alpha_\gamma e^{2\pi\gamma i}P^\gamma + \alpha_{\gamma'} e^{2\pi\gamma' i}P^{\gamma'};$$

$$\text{whence } e^{-2\pi ai}(\alpha_\beta e^{-2\pi\beta i}P^\beta + \alpha_{\beta'} e^{-2\pi\beta' i}P^{\beta'}) = \alpha_\gamma e^{2\pi\gamma i}P^\gamma + \alpha_{\gamma'} e^{2\pi\gamma' i}P^{\gamma'}. \quad (1)$$

In a manner precisely similar we obtain the equation

$$e^{-2\pi ai}(\alpha'_\beta e^{-2\pi\beta i}P^\beta + \alpha'_{\beta'} e^{-2\pi\beta' i}P^{\beta'}) = \alpha_\gamma e^{2\pi\gamma i}P^\gamma + \alpha'_{\gamma'} e^{2\pi\gamma' i}P^{\gamma'}. \quad (2)$$

Multiplying (1) by  $e^{-\sigma\pi i}$ ,  $\sigma$  being arbitrary, we find

$$e^{-\pi(2a+\sigma)i}(\alpha_\beta e^{-2\pi\beta i}P^\beta + \alpha_{\beta'} e^{-2\pi\beta' i}P^{\beta'}) = \alpha_\gamma e^{\pi i(2\gamma-\sigma)}P^\gamma + \alpha_{\gamma'} e^{\pi i(2\gamma'-\sigma)}P^{\gamma'}.$$

From this subtract  $\alpha_\gamma P^\gamma + \alpha_{\gamma'} P^{\gamma'} = \alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'}$  multiplied by  $e^{\sigma\pi i}$ , we thus obtain

$$\begin{aligned} \alpha_\beta P^\beta (-e^{\sigma\pi i} + e^{-\pi i(2a+2\beta+\sigma)}) + \alpha_{\beta'} P^{\beta'} (e^{-\pi i(2a+2\beta'+\sigma)} - e^{\sigma\pi i}) \\ = \alpha_\gamma P^\gamma (e^{\pi i(2\gamma-\sigma)} - e^{\sigma\pi i}) + \alpha_{\gamma'} P^{\gamma'} (e^{\pi i(2\gamma'-\sigma)} - e^{\sigma\pi i}). \end{aligned}$$

Remembering that  $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ , this becomes, omitting the factor  $2i$ ,

$$\begin{aligned} \alpha_\beta P^\beta e^{-\pi i(a+\beta)} \sin(\alpha + \beta + \sigma)\pi + \alpha_{\beta'} P^{\beta'} e^{-\pi i(a+\beta')} \sin(\alpha + \beta' + \sigma)\pi \\ = \alpha_\gamma P^\gamma e^{\pi i\gamma} \sin(\sigma - \gamma)\pi + \alpha_{\gamma'} P^{\gamma'} e^{\pi i\gamma'} \sin(\sigma - \gamma')\pi. \end{aligned} \quad (3)$$

In like manner we obtain from equation (2)

$$\begin{aligned} \alpha'_\beta P^\beta e^{-\pi i(a'+\beta)} \sin(\alpha' + \beta + \sigma)\pi + \alpha'_{\beta'} P^{\beta'} e^{-\pi i(a'+\beta')} \sin(\alpha' + \beta' + \sigma)\pi \\ = \alpha'_\gamma P^\gamma e^{\pi i\gamma} \sin(\sigma - \gamma)\pi + \alpha'_{\gamma'} P^{\gamma'} e^{\pi i\gamma'} \sin(\sigma - \gamma')\pi. \end{aligned} \quad (4)$$

We may so determine  $\sigma$  in each of equations (3) and (4) that one of the functions, say  $P^{\gamma'}$ , shall have its coefficient equal to 0.

To this end  $\sin(\sigma - \gamma')\pi$  must = 0; whence  $\sigma - \gamma'$  must = 0, 1, 2, . . . ; that is,  $\sigma$  must =  $\gamma'$  or differ from it by an integer only. Choosing  $\sigma = \gamma'$ , equations (3) and (4) become

$$\begin{aligned} \alpha_\beta P^\beta e^{-\pi i(a+\beta)} \sin(\alpha + \beta + \gamma')\pi + \alpha_{\beta'} P^{\beta'} e^{-\pi i(a+\beta')} \sin(\alpha + \beta' + \gamma')\pi \\ = \alpha_\gamma P^\gamma e^{\pi i\gamma} \sin(\gamma - \gamma')\pi, \end{aligned} \quad (5)$$

$$\begin{aligned} \alpha'_\beta P^\beta e^{-\pi i(a'+\beta)} \sin(\alpha' + \beta + \gamma')\pi + \alpha'_{\beta'} P^{\beta'} e^{-\pi i(a'+\beta')} \sin(\alpha' + \beta' + \gamma')\pi \\ = \alpha'_\gamma P^\gamma e^{\pi i\gamma} \sin(\gamma' - \gamma)\pi. \end{aligned} \quad (6)$$

Eliminating  $P^\gamma$  we find at last the homogeneous equation in  $P^\beta, P^{\beta'}$ :

$$\begin{aligned} \frac{\alpha_\beta}{\alpha_\gamma} P^\beta e^{-\pi i(\alpha+\beta)} \sin(\alpha+\beta+\gamma')\pi + \frac{\alpha_{\beta'}}{\alpha_\gamma} P^{\beta'} e^{-\pi i(\alpha+\beta')} \sin(\alpha+\beta'+\gamma')\pi \\ = \frac{\alpha'_\beta}{\alpha'_\gamma} P^\beta e^{-\pi i(\alpha'+\beta)} \sin(\alpha'+\beta+\gamma)\pi + \frac{\alpha'_{\beta'}}{\alpha'_\gamma} P^{\beta'} e^{-\pi i(\alpha'+\beta')} \sin(\alpha'+\beta'+\gamma)\pi. \end{aligned}$$

But since  $\beta$  is not equal to  $\beta'$ ,  $\frac{P^\beta}{P^{\beta'}}$  cannot be a constant, for

$$P^\beta = \left(\frac{1}{x}\right)^{-\beta} Y^\beta; P^{\beta'} = \left(\frac{1}{x}\right)^{-\beta'} Y^{\beta'}; Y^\beta \text{ and } Y^{\beta'}$$

being neither 0 nor  $\infty$  for  $x = \infty$ , which is the same as saying that each of them has a term independent of  $x$ .

Let  $c_\beta, c_{\beta'}$  be these terms, then

$$\frac{Y^\beta}{Y^{\beta'}} = \frac{c_\beta}{c_{\beta'}} + \dots \text{ and } \frac{P^\beta}{P^{\beta'}} = \left(\frac{1}{x}\right)^{-\beta+\beta'} \left(\frac{c_\beta}{c_{\beta'}} + \dots\right),$$

which cannot be a constant unless  $\beta = \beta'$ .

Hence the coefficients of  $P^\beta$  and  $P^{\beta'}$  must separately vanish and the following relations result:

$$\frac{\alpha_\gamma}{\alpha'_\gamma} = \frac{\alpha_\beta}{\alpha'_\beta} \frac{\sin(\alpha+\beta+\gamma')\pi}{\sin(\alpha'+\beta+\gamma)\pi} e^{\pi i(\alpha'-\alpha)} = \frac{\alpha_{\beta'}}{\alpha'_\beta} \frac{\sin(\alpha+\beta'+\gamma')\pi}{\sin(\alpha'+\beta'+\gamma)\pi} e^{\pi i(\alpha'-\alpha)}. \quad (7)$$

Again, if we eliminate  $P^\gamma$  we shall have  $\sigma = \gamma$ , and as before, or simply by interchanging  $\gamma'$  and  $\gamma$  in (7),

$$\frac{\alpha_\gamma}{\alpha'_\gamma} = \frac{\alpha_\beta}{\alpha'_\beta} \frac{\sin(\alpha+\beta+\gamma)\pi}{\sin(\alpha'+\beta+\gamma')\pi} e^{\pi i(\alpha'-\alpha)} = \frac{\alpha_{\beta'}}{\alpha'_\beta} \frac{\sin(\alpha+\beta+\gamma)\pi}{\sin(\alpha'+\beta'+\gamma')\pi} e^{\pi i(\alpha'-\alpha)}. \quad (8)$$

From (7) and (8) are obtained the two following values for the ratio  $\frac{\alpha_\beta}{\alpha'_\beta} : \frac{\alpha_{\beta'}}{\alpha'_\beta}$ :

$$\text{a) } \frac{\sin(\alpha+\beta'+\gamma')\pi \sin(\alpha'+\beta+\gamma)\pi}{\sin(\alpha'+\beta'+\gamma)\pi \sin(\alpha+\beta+\gamma')\pi}, \text{ and b) } \frac{\sin(\alpha+\beta'+\gamma)\pi \sin(\alpha'+\beta+\gamma')\pi}{\sin(\alpha'+\beta'+\gamma')\pi \sin(\alpha+\beta+\gamma)\pi}.$$

But since  $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$ , the first value may be written

$$\frac{\sin(1-\alpha'-\beta-\gamma)\pi \sin(1-\alpha-\beta'-\gamma)\pi}{\sin(1-\alpha-\beta-\gamma)\pi \sin(1-\alpha'-\beta'-\gamma)\pi};$$

and remembering that  $\sin(\pi - \theta) = \sin \theta$ , this is seen to be the same as the second value.

The four relations in (7) and (8) are all included in the symbolic expression

$$CBA = 1,$$

which in fact we actually employed under the form

$$BA = C^{-1}.$$

Having thus three of the ratios

$$\frac{\alpha_\gamma}{\alpha'_\gamma}, \frac{\alpha_\beta}{\alpha'_\beta}, \frac{\alpha_{\beta'}}{\alpha'_{\beta'}}, \frac{\alpha_{\gamma'}}{\alpha'_{\gamma'}}$$

expressed in terms of the fourth, it is apparent that three of the quantities  $\alpha_\gamma, \alpha_{\gamma'}, \alpha'_\gamma, \alpha'_{\gamma'}, \alpha_\beta, \alpha_{\beta'}, \alpha'_\beta, \alpha'_{\beta'}$  may be expressed in terms of the remaining five. Clearly there are no more relations between them, for  $CBA = 1$  applied to each of the equations

$$\left. \begin{aligned} P^\alpha &= \alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'}, \\ P^{\alpha'} &= \alpha'_\beta P^\beta + \alpha'_{\beta'} P^{\beta'}, \\ P^\alpha &= \alpha_\gamma P^\gamma + \alpha_{\gamma'} P^{\gamma'}, \\ P^{\alpha'} &= \alpha'_\gamma P^\gamma + \alpha'_{\gamma'} P^{\gamma'}, \end{aligned} \right\} \quad (9)$$

can give one, and only one, relation.

But from the relation

$$P = c_\alpha P^\alpha + c_{\alpha'} P^{\alpha'},$$

where  $c_\alpha$  and  $c_{\alpha'}$  are arbitrary, together with the relations (9), when a particular value is assigned to  $x$ , they may be completely determined, and so determined that each shall be finite.

If  $P_1$  is a function with the same exponents as  $P$ , then by a proper choice of the initial values and arbitrary constants we may make any selected five of the quantities

$$\alpha_\beta, \alpha_{\beta'}, \alpha_\gamma, \alpha_{\gamma'}, \alpha'_\beta, \alpha'_{\beta'}, \alpha'_\gamma, \alpha'_{\gamma'}$$

the same in each. The remaining three will then, as already seen, be the same in each, and we shall have the following identical relations:

$$\begin{aligned} P^\alpha &= \alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'}, & P^{\alpha'} &= \alpha'_\beta P^\beta + \alpha'_{\beta'} P^{\beta'}, \\ P_1^\alpha &= \alpha_\beta P_1^\beta + \alpha_{\beta'} P_1^{\beta'}, & P_1^{\alpha'} &= \alpha'_\beta P_1^\beta + \alpha'_{\beta'} P_1^{\beta'}, \end{aligned}$$

whence

$$\left| \begin{array}{c} P^\alpha, P_1^\alpha \\ P^{\alpha'}, P_1^{\alpha'} \end{array} \right| = \left| \begin{array}{c} \alpha_\beta, \alpha_{\beta'} \\ \alpha'_\beta, \alpha'_{\beta'} \end{array} \right| \left| \begin{array}{c} P^\beta, P_1^\beta \\ P^{\beta'}, P_1^{\beta'} \end{array} \right|$$

identically; and, in precisely the same way,

$$\left| \begin{array}{c} P^\alpha, P_1^\alpha \\ P^{\alpha'}, P_1^{\alpha'} \end{array} \right| = \left| \begin{array}{c} \alpha_\gamma, \alpha_{\gamma'} \\ \alpha'_\gamma, \alpha'_{\gamma'} \end{array} \right| \left| \begin{array}{c} P^\gamma, P_1^\gamma \\ P^{\gamma'}, P_1^{\gamma'} \end{array} \right|$$



We notice that if  $(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha)$  be multiplied by  $x^{-\alpha-\alpha'}$  it becomes a uniform function in the region of  $x=0$ , which is neither 0 nor  $\infty$  for  $x=0$ . The same is true for  $(P^\beta P_1^{\beta'} - P^{\beta'} P_1^\beta) \left(\frac{1}{x}\right)^{-\beta-\beta'}$  for  $x=\infty$ ; and of

$$(P^\gamma P_1^{\gamma'} - P^{\gamma'} P_1^\gamma)(1-x)^{-\gamma-\gamma'} \text{ for } x=1.$$

Moreover, 0,  $\infty$ , 1 are the only branch points. Writing therefore

$$(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'} = (\alpha_\gamma \alpha_{\gamma'} - \alpha_{\gamma'} \alpha_\gamma) (P^\gamma P_1^{\gamma'} - P^{\gamma'} P_1^\gamma) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'} \quad (10)$$

the first member is clearly uniform and continuous at the point 0; the second at the point 1; hence both members are uniform and continuous in the region of 0 and 1. But  $(P^\beta P_1^{\beta'} - P^{\beta'} P_1^\beta) x^{\beta+\beta'}$  is uniform and continuous in the region of  $\infty$ ; the same is therefore true of  $(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'}$  which, for  $x=\infty$ , contains the factor  $x^{-\alpha-\alpha'-\gamma-\gamma'} = x^{\beta+\beta'-1}$ . Consequently,

$$(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'}$$

has no singular points. It is therefore a constant.

Recurring to the identical relation

$$(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'} = (\alpha_\beta \alpha_{\beta'} - \alpha_{\beta'} \alpha_\beta) (P^\beta P_1^{\beta'} - P^{\beta'} P_1^\beta) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'}, \quad (11)$$

we observe that when mod  $x$  is very large this reduces to

$$(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha) x^{\beta+\beta'-1} = (\alpha_\beta \alpha_{\beta'} - \alpha_{\beta'} \alpha_\beta) (P^\beta P_1^{\beta'} - P^{\beta'} P_1^\beta) x^{\beta+\beta'-1},$$

of which the second member vanishes, for  $x=\infty$ , since  $(P^\beta P_1^{\beta'} - P^{\beta'} P_1^\beta) x^{\beta+\beta'}$  is finite, for  $x=\infty$ . Hence  $(P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha) x^{-\alpha-\alpha'} (1-x)^{-\gamma-\gamma'}$  vanishes, for  $x=\infty$ . Its value is therefore always 0, and the inference is immediate that  $P^\alpha P_1^{\alpha'} - P^{\alpha'} P_1^\alpha = 0$ : whence

$$\frac{P_1^{\alpha'}}{P^{\alpha'}} = \frac{P_1^\alpha}{P^\alpha}.$$

In the same way we find

$$\frac{P_1^{\beta'}}{P^{\beta'}} = \frac{P_1^\beta}{P^\beta} = \frac{\alpha_\beta P_1^\beta + \alpha_{\beta'} P_1^{\beta'}}{\alpha_\beta P^\beta + \alpha_{\beta'} P^{\beta'}} = \frac{P_1^\beta}{P^\beta},$$

the third member being obtained by combining the ratios. Likewise

$$\frac{P_1^{\gamma'}}{P^{\gamma'}} = \frac{P_1^\gamma}{P^\gamma} = \frac{\alpha_\gamma P_1^\gamma + \alpha_{\gamma'} P_1^{\gamma'}}{\alpha_\gamma P^\gamma + \alpha_{\gamma'} P^{\gamma'}} = \frac{P_1^\gamma}{P^\gamma}.$$

Now  $\frac{P_1^{\alpha'}}{P^\alpha}$  is uniform and continuous at the point 0;  $\frac{P_1^{\beta'}}{P^\beta}$  at the point  $\infty$ , and  $\frac{P_1^{\gamma'}}{P^\gamma}$  at the point 1; hence  $\frac{P_1^{\alpha}}{P^\alpha}$  is everywhere uniform and continuous; unless for some value of  $x$  other than 0,  $\infty$  or 1,  $P^\alpha$  and  $P^{\alpha'}$  both vanish, in which case all these relations become illusory. But this cannot happen, for by aid of equations (9) we may write

$$P^\alpha \frac{dP^{\alpha'}}{dx} - P^{\alpha'} \frac{dP^\alpha}{dx} = M \left( P^\beta \frac{dP^{\beta'}}{dx} - P^{\beta'} \frac{dP^\beta}{dx} \right) = N \left( P^\gamma \frac{dP^{\gamma'}}{dx} - P^{\gamma'} \frac{dP^\gamma}{dx} \right),$$

where

$$M = \begin{vmatrix} \alpha_\beta & \alpha'_\beta \\ \alpha_{\beta'} & \alpha'_{\beta'} \end{vmatrix}, \quad N = \begin{vmatrix} \alpha_\gamma & \alpha'_\gamma \\ \alpha_{\gamma'} & \alpha'_{\gamma'} \end{vmatrix}.$$

And since

$$P^\alpha P^{\alpha'} x^{-\alpha-\alpha'}$$

does not vanish for  $x=0$ , it follows that  $P^\alpha P^{\alpha'}$  is zero of the order  $\alpha + \alpha'$  for  $x=0$ , and consequently that

$$P^\alpha \frac{dP^{\alpha'}}{dx} - P^{\alpha'} \frac{dP^\alpha}{dx}$$

is zero of order  $\alpha + \alpha' - 1$  for  $x=0$ .

Likewise  $P^\beta \frac{dP^{\beta'}}{dx} - P^{\beta'} \frac{dP^\beta}{dx}$  is zero of order  $\beta + \beta' + 1$  for  $\frac{1}{x} = 0$ .

For  $P^\beta P^{\beta'}$  contains  $\frac{1}{x}$  to the power  $\beta + \beta'$ , when  $\text{mod } x$  is large, and differentiating we introduce the factor  $\frac{1}{x}$  once more. Finally,  $P^\gamma \frac{dP^{\gamma'}}{dx} - P^{\gamma'} \frac{dP^\gamma}{dx}$  is zero of order  $\gamma + \gamma' - 1$  for  $x=1$ .

Reasoning now precisely as upon equations (10) and (11), we find that

$$\left( P^\alpha \frac{dP^{\alpha'}}{dx} - P^{\alpha'} \frac{dP^\alpha}{dx} \right) x^{-\alpha-\alpha'+1} (1-x)^{-\gamma-\gamma'+1}$$

is everywhere uniform and continuous, and therefore a constant. If its value were 0 we should have  $\frac{1}{P^\alpha} \frac{dP^\alpha}{dx} = \frac{1}{P^{\alpha'}} \frac{dP^{\alpha'}}{dx}$ , that is,

$$\log P^\alpha = \log P^{\alpha'} + \text{cons.} \quad (12)$$

But  $P^\alpha = x^\alpha Y^\alpha$ ;  $P^{\alpha'} = x^{\alpha'} Y^{\alpha'}$ , and equation (12) leads us to the following,

$$\alpha \log x - \alpha' \log x + \log Y^\alpha - \log Y^{\alpha'} = \text{cons.},$$

which must hold for any value of  $\log x$  whatever. This can only be the case if  $\alpha = \alpha'$ , which is contrary to hypothesis.

If, now,  $P^\alpha$  and  $P^{\alpha'}$  were simultaneously 0 for any value of  $x$  other than 0,  $\infty$ , 1, the value of the constant

$$\left( P^\alpha \frac{dP^{\alpha'}}{dx} - P^{\alpha'} \frac{dP^\alpha}{dx} \right) x^{-\alpha-\alpha'+1} (1-x)^{-\gamma-\gamma'+1}$$

would necessarily be zero; hence  $P^\alpha$  and  $P^{\alpha'}$  cannot so vanish. We therefore infer that  $\frac{P_1^\alpha}{P^\alpha}$  is a constant.

We are thus led to the theorem:

If two *P*-functions have the same exponents, the branches of each corresponding to the same exponent can differ only by a constant factor.

We have then

$$\frac{P_1^\alpha}{P^\alpha} = \frac{P_1^{\alpha'}}{P^{\alpha'}} = \frac{P_1^\beta}{P^\beta} = \frac{P_1^{\beta'}}{P^{\beta'}} = \frac{P_1^\gamma}{P^\gamma} = \frac{P_1^{\gamma'}}{P^{\gamma'}} = g, \text{ a constant.}$$

Therefore

$$P_1^\alpha = gP^\alpha; \quad P_1^{\alpha'} = gP^{\alpha'}$$

and

$$P_1 = c_\alpha g P^\alpha + c_{\alpha'} g P^{\alpha'} = c_1 P^\alpha + c_2 P^{\alpha'};$$

that is, as was previously observed, Riemann's definition determines the *P*-function to within two arbitrary constants.

Recurring to equations (7) and (8), it may be noticed that the numerators differ from the denominators only by containing  $\alpha$  instead of  $\alpha'$ . Therefore it is evident that to increase or decrease  $\beta, \beta'; \gamma, \gamma'$  by any integers whatever cannot alter the values of the ratios. As to  $\alpha, \alpha'$ , if one of them be increased or decreased by an odd and the other by an even integer, the fraction will change sign, but  $e^{\pi i(\alpha-\alpha')}$  will change sign at the same time. If both  $\alpha$  and  $\alpha'$  be increased by odd or even integers, there will be no change of sign in either factor. Hence, to alter the exponents of the *P*-functions by any integers whatever will not alter these ratios. Therefore, if in two *P*-functions, whose exponents differ only by integers, we assume the five arbitrary quantities  $\alpha_\beta, \alpha_{\beta'}, \alpha'_\beta, \alpha'_{\beta'}, \alpha_\gamma$  the same in each, as we may, then the remaining three  $\alpha_\gamma, \alpha'_\gamma, \alpha'_{\gamma'}$  as determined by equations (7) and (8), will be the same in each.

Calling the two functions  $P \left( \begin{smallmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{smallmatrix} x \right)$  and  $P_1 \left( \begin{smallmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha'_1 & \beta'_1 & \gamma'_1 \end{smallmatrix} x \right)$  we shall have from the equations

$$\left. \begin{aligned} P^{\alpha} &= \alpha_{\beta} P^{\beta} + \alpha_{\beta'} P^{\beta'} = \alpha_{\gamma} P^{\gamma} + \alpha_{\gamma'} P^{\gamma'}, \\ P^{\alpha_1} &= \alpha'_{\beta} P^{\beta} + \alpha'_{\beta'} P^{\beta'} = \alpha_{\gamma} P^{\gamma} + \alpha'_{\gamma'} P^{\gamma'}, \\ P^{\alpha_1} &= \alpha_{\beta} P^{\beta_1} + \alpha_{\beta'} P^{\beta'_1} = \alpha_{\gamma} P^{\gamma_1} + \alpha_{\gamma'} P^{\gamma'_1}, \\ P^{\alpha'_1} &= \alpha'_{\beta} P^{\beta_1} + \alpha'_{\beta'} P^{\beta'_1} = \alpha'_{\gamma} P^{\gamma_1} + \alpha'_{\gamma'} P^{\gamma'_1}, \end{aligned} \right\} \quad (13)$$

the resulting equation

$$P^{\alpha} P^{\alpha'_1} - P^{\alpha'} P^{\alpha_1} = M (P^{\beta} P^{\beta'_1} - P^{\beta'} P^{\beta_1}) = N (P^{\gamma} P^{\gamma'_1} - P^{\gamma'} P^{\gamma_1}). \quad (14)$$

For greater clearness, suppose that  $\alpha + \alpha'_1, \beta + \beta'_1, \gamma + \gamma'_1$  exceed  $\alpha' + \alpha_1, \beta' + \beta_1, \gamma' + \gamma_1$  respectively, by positive integers. Then by considering the first and third members of (14) we conclude that

$$(P^{\alpha} P^{\alpha'_1} - P^{\alpha'} P^{\alpha_1}) x^{-\alpha' - \alpha_1} (1 - x)^{-\gamma' - \gamma_1}$$

is uniform and continuous in the regions of the points 0 and 1 and for all finite values of  $x$ ; and, by considering the second member, we find that it is infinite of the order  $-\alpha' - \alpha_1 - \gamma' - \gamma_1 - \beta' - \beta_1$  for  $x = \infty$ . It is therefore an entire function of  $x$  of degree  $-\alpha' - \alpha_1 - \gamma' - \gamma_1 - \beta' - \beta_1$ , which number is an integer. Designate this function by  $F$ . Now

$$\begin{aligned} \alpha - \alpha' + \beta - \beta' + \gamma - \gamma' &= \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' - 2(\alpha' + \beta' + \gamma') \\ &= 1 - 2(\alpha' + \beta' + \gamma'). \end{aligned}$$

Hence, when  $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$  are altered by integers, the sum of the differences  $\alpha - \alpha' = \lambda; \beta - \beta' = \mu; \gamma - \gamma' = \nu$  changes by twice the sum of increments of  $\alpha', \beta'$  and  $\gamma'$ ; that is, by an even number. Put also

$$\alpha_1 - \alpha'_1 = \lambda_1, \beta_1 - \beta'_1 = \mu_1, \gamma_1 - \gamma'_1 = \nu_1,$$

and let  $\Delta\lambda, \Delta\mu, \Delta\nu$  designate the absolute values of

$$\lambda - \lambda_1, \mu - \mu_1, \nu - \nu_1,$$

and, to fix the ideas, suppose as before that  $\alpha + \alpha'_1, \beta + \beta'_1, \gamma + \gamma'_1$  exceed  $\alpha_1 + \alpha', \beta_1 + \beta', \gamma_1 + \gamma'$  by positive integers respectively, then

$$\alpha_1 + \alpha' = \frac{\alpha + \alpha'_1 + \alpha_1 + \alpha'}{2} - \frac{\Delta\lambda}{2}.$$

Hence we see that

$$\begin{aligned} -\alpha' - \alpha_1 &= -\frac{\alpha + \alpha'_1 + \alpha' + \alpha_1}{2} + \frac{\Delta\lambda}{2} \\ -\beta' - \beta_1 &= -\frac{\beta + \beta'_1 + \beta' + \beta_1}{2} + \frac{\Delta\mu}{2} \\ -\gamma' - \gamma_1 &= -\frac{\gamma + \gamma'_1 + \gamma' + \gamma_1}{2} + \frac{\Delta\nu}{2}. \end{aligned}$$

Hence the degree of the function  $F$  is  $\frac{\Delta\lambda + \Delta\mu + \Delta\nu}{2} - 1$ . Furthermore, if

$P\left(\begin{smallmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{smallmatrix} x\right)$ ,  $P_1\left(\begin{smallmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha'_1, & \beta'_1, & \gamma'_1 \end{smallmatrix} x\right)$ ,  $P_2\left(\begin{smallmatrix} \alpha_2, & \beta_2, & \gamma_2 \\ \alpha'_2, & \beta'_2, & \gamma'_2 \end{smallmatrix} x\right)$  are three  $P$ -functions whose exponents differ only by integers, we observe, by what precedes, that in the identical equation

$$P^2(P_1'P_2' - P_1'P_2') + P_1'(P_2'P_1' - P_2'P_1') + P_2'(P_1'P_2' - P_1'P_2') = 0,$$

the coefficients of  $P^2$ ,  $P_1'$  and  $P_2'$  are entire functions of  $x$ .

But

$$P = c_1 x^\alpha Y^{(\alpha)} + c_2 x^{\alpha'} Y^{(\alpha')}$$

in the region of the point  $x = 0$ . Hence, in the region of the point  $x = 0$ ,

$$\frac{dP}{dx} = c_1 x^{\alpha-1} (\alpha Y^{(\alpha)} + x Y'^{(\alpha)}) + c_2 x^{\alpha'-1} (\alpha' Y^{(\alpha')} + x Y'^{(\alpha')})$$

which has evidently  $\alpha - 1$  and  $\alpha' - 1$  for exponents.

In this way we see that the exponents of  $P$ ,  $\frac{dP}{dx}$  and  $\frac{d^2P}{dx^2}$  differ only by integers, hence an identical relation in which the coefficients are rational functions of  $x$  exists between any  $P$ -function and its first and second differential coefficients. In other words, the  $P$ -function satisfies a linear differential equation of the second order.

## PART II.

### THE DIFFERENTIAL EQUATION SATISFIED BY THE $P$ -FUNCTION.

#### SECTION 1.—*The Properties of the P-function stated as Properties of an Integral.*

1.  $P$  is a regular integral of its differential equation. This results from the fact that  $g$  being any one of the singular points of  $P$ , it has in the region of  $g$  the form

$$P = c_1 (x - g)^\alpha Y^{(\alpha)} + c_2 (x - g)^{\alpha'} Y^{(\alpha')},$$

$Y^{(\alpha)}$  and  $Y^{(\alpha')}$  being neither zero nor infinite for  $x = g$ , and this is by definition the characteristic of a regular integral.

2. The quantities  $\alpha, \alpha'; \beta, \beta'$  and  $\gamma, \gamma'$  are the roots of the indicial equations in the regions of the points  $a, b$  and  $c$  respectively. For these roots are the negative exponents of the factors by which the integrals in the regions of

those points must be multiplied to make them uniform, finite and continuous and not zero at the singular points; and this is the property of the exponents  $\alpha, \alpha'$  in the region of  $x = a$ ;  $\beta, \beta'$  in the region of  $b$ , and  $\gamma, \gamma'$  in the region of  $c$ .

3. The points  $a, b$  and  $c$  are critical points of the coefficients of the equation, because only the critical points of the coefficients can be branch points of the integrals; and because the coefficients are rational, these critical points must be poles.

Also, it is possible to determine a differential equation whose coefficients have no other critical points than  $a, b, c$ , of which the  $P$ -function is the general integral. Hence, any other equation having coefficients with these and other different critical points, and  $P$  for an integral, will not be irreducible.

4. Since none of the differences  $\alpha - \alpha', \beta - \beta', \gamma - \gamma'$  is an integer, the integral  $P$  is free from logarithms. The familiar properties of the regular linear differential equation here utilized will be found stated in Craig's Linear Differential Equations.

#### SECTION 2.—Theorem.

The indicial equation corresponding to any pole of the coefficients of a linear differential equation of the second order, with regular integrals, is not altered when the position of the pole is changed by a linear transformation of the form

$$x = \frac{-hx' + f}{gx' - e}.$$

Let

$$\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0 \quad (1)$$

be the equation in question.

Putting  $x = \frac{-hx' + f}{gx' - e}$  and making the necessary computations, we find

$$\begin{aligned} \frac{dy}{dx} &= -\frac{(gx' - e)^2}{gf - he} \cdot \frac{dy}{dx'}, \\ \frac{d^2y}{dx^2} &= \frac{(gx' - e)^4}{(gf - he)^2} \cdot \frac{d^2y}{dx'^2} + \frac{2g(gx' - e)^3}{(gf - he)^2} \cdot \frac{dy}{dx'}. \end{aligned}$$

Since the integrals are regular, we notice that

$$p_1 = \frac{\phi(x)}{x - a} = \frac{gx' - e}{ga + h} \cdot \frac{\phi(x)}{a' - x'},$$

where  $a' = \frac{f + ae}{h + ag}$ , and



$$p_2 = \frac{\psi(x)}{(x-a)^2} = \frac{(gx' - e)^2}{(ga + h)^2} \cdot \frac{\psi(x)}{(a' - x')^2}.$$

Hence equation (1) becomes

$$\begin{aligned} \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left[ \frac{2g}{gx' - e} - \frac{gf - he}{(gx' - e)(ga + h)} \cdot \frac{\psi(x)}{a' - x'} \right] \\ + \frac{(gf - he)^2}{(ga + h)^2} \cdot \frac{1}{(gx' - e)^2} \cdot \frac{\psi(x)}{(a' - x')^2} \cdot y = 0, \quad (2) \end{aligned}$$

observing that  $x - a = (a' - x') \frac{h + ag}{gx' - e}$ .

If now  $(x - a)p_1 = a_0$  and  $(x - a)^2 p_2 = b_0$ , when  $x = a$ , the indicial equation for the point  $a$  is

$$r(r - 1) + a_0 r + b_0 = 0 = r(r - 1) + r\phi(a) + \psi(a).$$

Calling  $q_1$  and  $q_2$  the coefficients of (2), we must find the values of  $(x' - a')q_1$  and  $(x' - a')^2 q_2$  when  $x' = a'$ , since  $a'$  is the new pole, and  $x' = a'$  when  $x = a$ .

We have

$$(x' - a')q_1 = \frac{gf - he}{ga' - e} \phi(a) \frac{1}{ga + h},$$

$$(x' - a')^2 q_2 = \frac{(gf - he)^2}{(ga' - e)^2} \frac{\psi(a)}{(ga + h)^2},$$

when  $x' = a'$ .

But

$$ga' - e = \frac{gf - he}{ag + h};$$

hence

$$(x' - a')q_1 = \phi(a),$$

$$(x' - a')^2 q_2 = \psi(a),$$

when  $x' = a'$ . Hence the new indicial equation is the same as the old one.

SECTION 3.—*The Transformation*  $x = \frac{-hx' + f}{gx' - e}$ , or  $x' = \frac{ex + f}{gx + h}$ .

When

$$x = a, \quad x' = \frac{ea + f}{ga + h} = a',$$

$$x = b, \quad x' = \frac{eb + f}{gb + h} = b',$$

$$x = c, \quad x' = \frac{ec + f}{gc + h} = c',$$

$$\frac{1}{x - a} = \frac{gx' - e}{ga + h} \cdot \frac{1}{a' - x'},$$

$$\frac{1}{x - b} = \frac{gx' - e}{gb + h} \cdot \frac{1}{b' - x'},$$

$$\frac{1}{x - c} = \frac{gx' - e}{gc + h} \cdot \frac{1}{c' - x'};$$

hence the poles are merely changed in position by this transformation, and by assigning suitable values to the constants we may place them at pleasure in the plane.

1. If

$$\begin{aligned} f &= -ae, \\ h &= -gb, \\ e(c-a) &= g(c-b), \text{ or } g = \frac{e(c-a)}{c-b}, \end{aligned}$$

then

$$a' = 0, \quad b' = \infty, \quad c' = 1,$$

and

$$x' = \frac{e(x-a)}{g(x-b)}.$$

2. Taking  $e = 1$ , these results are simplified,

$$\begin{aligned} f &= -a, \\ g &= \frac{c-a}{c-b}, \\ h &= -b \frac{c-a}{c-b}, \\ a' &= 0, \quad b' = \infty, \quad c' = 1, \\ x' &= \frac{c-b}{c-a} \cdot \frac{x-a}{x-b}. \end{aligned}$$

Thus, by this transformation, without changing the indicial equations, and therefore without changing the exponents  $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ , we may place the branch points of the integral  $P$  at 0,  $\infty$  and 1. Any one of the branch points 0,  $\infty$  and 1 may be made to correspond to any pair of exponents  $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ . For the possible arrangements are

- (1) 0,  $\infty$ , 1,
- (2) 0, 1,  $\infty$ ,
- (3) 1,  $\infty$ , 0,
- (4) 1, 0,  $\infty$ ,
- (5)  $\infty$ , 0, 1,
- (6)  $\infty$ , 1, 0.

For (2) we must take

$$e = 1, \quad f = -a, \quad b-a = gb + h, \quad gc + h = 0,$$

or

$$h = -cg; \quad b-a = g(b-c); \quad g = \frac{b-a}{b-c}; \quad h = -c \frac{b-a}{b-c} \cdot x'' = \frac{x-a}{x-c} \cdot \frac{b-c}{b-a}.$$

For (3),

$$\begin{aligned} a + f &= ga + h, \\ gb + h &= 0, \\ c + f &= 0, \\ a - c &= g(a - b), \text{ or } g = \frac{a - c}{a - b}; f = -c, h = -b \frac{a - c}{a - b}. x^{\text{III}} = \frac{x - c}{x - b} \cdot \frac{a - b}{a - c}. \end{aligned}$$

For (4),

$$\begin{aligned} a + f &= ga + h, \\ b + f &= 0, \\ gc + h &= 0, \\ a - b &= g(a - c); g = \frac{a - b}{a - c}; f = -b; h = -c \frac{a - b}{a - c}. x^{\text{IV}} = \frac{x - b}{x - c} \cdot \frac{a - c}{a - b}. \end{aligned}$$

For (5),

$$\begin{aligned} ga + h &= 0, \\ b + f &= 0, \\ c + f &= gc + h, \\ c - b &= g(c - a); g = \frac{c - b}{c - a}; f = -b; h = -a \frac{c - b}{c - a}. x^{\text{V}} = \frac{x - b}{x - a} \cdot \frac{c - a}{c - b}. \end{aligned}$$

For (6),

$$\begin{aligned} ga + h &= 0, \\ b + f &= gb + h, \\ c + f &= 0, \\ b - c &= g(b - a); g = \frac{b - c}{b - a}; f = -c; h = -a \frac{b - c}{b - a}. x^{\text{VI}} = \frac{x - c}{x - a} \cdot \frac{b - a}{b - c}. \end{aligned}$$

These variables satisfy the following equations:

$$\begin{aligned} x^{\text{V}} &= \frac{1}{x^{\text{I}}}, \\ x^{\text{VI}} &= \frac{1}{x^{\text{II}}}, \\ x^{\text{III}} &= \frac{1}{x^{\text{IV}}}, \\ x^{\text{II}} x^{\text{III}} &= -x^{\text{I}}, \\ x^{\text{III}} - 1 &= -x^{\text{I}}. \end{aligned}$$

Whence

$$\begin{aligned} x^{\text{vi}} &= -\frac{x' - 1}{x'}, \\ x^{\text{v}} &= \frac{1}{x'}, \\ x^{\text{iv}} &= \frac{1}{1 - x'}, \\ x^{\text{iii}} &= 1 - x', \\ x^{\text{ii}} &= -\frac{x'}{1 - x'}. \end{aligned}$$

Thus all of them may be unambiguously expressed in terms of  $x'$ , and we may obtain the six forms of  $P$  mentioned in Section 3, Part I.

SECTION 4.—*Effect upon the Indicial Equation when the Integral is multiplied by a Factor of the Form  $(x - a)^{s_1} (x - b)^{s_2} (x - c)^{s_3}$ .*

Let the differential equation be

$$\frac{d^2 y}{dx^2} + \frac{\phi(x)}{x - a} \frac{dy}{dx} + \frac{\psi(x)}{(x - a)^2} y = 0 = \frac{d^2 y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y, \quad (1)$$

where only the pole  $x = a$  is brought into evidence; and let  $Y$  be an integral in the region of the point  $a$ . The corresponding indicial equation is

$$\begin{aligned} r(r - 1) + r\phi(a) + \psi(a) &= 0, \\ \text{or} \quad r^2 + r[\phi(a) - 1] + \psi(a) &= 0. \end{aligned}$$

If  $\alpha$  and  $\alpha'$  are the roots, then

$$\begin{aligned} \alpha + \alpha' &= 1 - \phi(a), \\ \alpha\alpha' &= \psi(a). \end{aligned}$$

If, now, the equation satisfied by

$$(x - a)^{s_1} (x - b)^{s_2} (x - c)^{s_3} y = y'$$

is

$$\frac{d^2 y'}{dx^2} + q_1 \frac{dy'}{dx} + q_2 y' = 0, \quad (2)$$

and if

$$(x - a)^{s_1} (x - b)^{s_2} (x - c)^{s_3} y = Ay,$$

then

$$\frac{d^2 y'}{dx^2} = A \frac{d^2 y}{dx^2} + 2A \frac{dy}{dx} \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right) - Ay \left( \frac{\delta_1}{(x-a)^2} + \frac{\delta_2}{(x-b)^2} + \frac{\delta_3}{(x-c)^2} \right) + Ay \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right)^2,$$

and

$$\frac{dy'}{dx} = A \frac{dy}{dx} + Ay \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right).$$

Thus equation (2) becomes, after dividing out the factor  $A$ ,

$$\left. \begin{aligned} & \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left[ q_1 + 2 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right) \right] \\ & + y \left[ \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right)^2 - \frac{\delta_1}{(x-a)^2} - \frac{\delta_2}{(x-b)^2} - \frac{\delta_3}{(x-c)^2} \right] \\ & + q_1 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right) + q_2 \end{aligned} \right\} = 0. \quad (3)$$

This can be no other than the equation

$$\frac{d^2 y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0,$$

and therefore, identifying the coefficients,

$$\begin{aligned} p_1 &= q_1 + 2 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right), \\ p_2 &= \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right)^2 - \frac{\delta_1}{(x-a)^2} - \frac{\delta_2}{(x-b)^2} - \frac{\delta_3}{(x-c)^2} \\ &\quad + q_1 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right) + q_2; \end{aligned}$$

that is,

$$\begin{aligned} q_1 &= p_1 - 2 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right), \\ q_2 &= p_2 - \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right)^2 + \frac{\delta_1}{(x-a)^2} + \frac{\delta_2}{(x-b)^2} + \frac{\delta_3}{(x-c)^2} \\ &\quad - q_1 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} + \frac{\delta_3}{x-c} \right). \end{aligned}$$

Hence  $a$  is a pole of the same degree of multiplicity for the new coefficients as

for the old. To form the indicial equation we calculate  $(x-a)q_1$  and  $(x-a)^2q_2$  for  $x=a$ ,

$$\begin{aligned}(x-a)q_1 &= \phi(a) - 2\delta_1, \\ (x-a)^2q_2 &= \psi(a) - \delta_1^2 + \delta_1 - \delta_1[\phi(a) - 2\delta_1],\end{aligned}$$

and the equation is

$$r^2 + r[\phi(a) - 2\delta_1 - 1] + \psi(a) - \delta_1\phi(a) + \delta_1^2 + \delta_1 = 0.$$

If the roots of this be  $\sigma$  and  $\sigma'$ , then

$$\begin{aligned}\sigma + \sigma' &= 1 + 2\delta_1 - \phi(a), \\ \sigma\sigma' &= \psi(a) - \delta_1\phi(a) + \delta_1 + \delta_1^2.\end{aligned}$$

Hence

$$\begin{aligned}\sigma + \sigma' &= \alpha + \alpha' + 2\delta_1 = (\alpha + \delta_1) + (\alpha' + \delta_1) \\ \sigma\sigma' &= \alpha\alpha' + \delta_1(\alpha + \alpha') + \delta_1^2 = (\alpha + \delta_1)(\alpha' + \delta_1).\end{aligned}$$

Whence we conclude that

$$\begin{aligned}\sigma &= \alpha + \delta_1, \\ \sigma' &= \alpha' + \delta_1;\end{aligned}$$

therefore, to multiply the integral of equation (1) by the factor

$$(x-a)^{\delta_1}(x-b)^{\delta_2}(x-c)^{\delta_3}$$

increases the roots of the indicial equation corresponding to each pole by the exponent of the corresponding factor, but leaves their difference unchanged.

Suppose the point infinity is a singular point of the integral of equation (1), the other singular points being  $a$  and  $b$ . If we multiply  $Y$ , the integral, by  $(x-a)^{\delta_1}(x-b)^{\delta_2}$ , the roots of the indicial equations at  $a$  and  $b$  will be increased by  $\delta_1$  and  $\delta_2$  respectively, as we have seen, and the transformed equation (3) becomes

$$\left. \begin{aligned} \frac{d^2y}{dx^2} + \frac{dy}{dx} \left[ q_1 + 2 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} \right) \right] \\ + y \left[ \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} \right)^2 - \frac{\delta_1}{(x-a)^2} - \frac{\delta_2}{(x-b)^2} \right] \\ + q_1 \left( \frac{\delta_1}{x-a} + \frac{\delta_2}{x-b} \right) + q_2 \end{aligned} \right\} = 0. \quad (4)$$

In equation (1) let us make the transformation

$$x = \frac{1}{x'} \text{ or } x' = \frac{1}{x}.$$



Whence

$$\frac{dy}{dx} = -x'^2 \frac{dy}{dx'}$$

$$\frac{d^2y}{dx^2} = 2x' \frac{dy}{dx'} + x'^4 \frac{d^2y}{dx'^2},$$

and equation (1) becomes

$$\frac{d^2y}{dx^2} + \frac{dy}{dx'} \left[ \frac{2}{x'} - \frac{p'_1}{x'^2} \right] + \frac{p'_2}{x'^4} y = 0. \quad (5)$$

Of this equation,  $x' = 0$  must be a pole, by hypothesis, and the corresponding indicial equation is, if

$$\frac{p'_1}{x'^2} = \frac{\phi(x')}{x'}; \quad \frac{p'_2}{x'^4} = \frac{\psi(x')}{x'^2},$$

$$r^2 + r[\phi(0) + 1] + \psi(0) = 0. \quad (6)$$

Again,

$$q'_1 = p'_1 - 2 \frac{\delta_1 x'}{1 - ax'} - 2 \frac{\delta_2 x'}{1 - bx'},$$

$$q'_2 = p'_2 - \left[ \left( \frac{\delta_1}{1 - ax'} + \frac{\delta_2}{1 - bx'} \right)^2 - \frac{\delta_1}{(1 - ax')^2} - \frac{\delta_2}{(1 - bx')^2} \right] x'^2$$

$$- q'_1 x' \left( \frac{\delta_1}{1 - ax'} + \frac{\delta_2}{1 - bx'} \right).$$

Transforming (2) by the substitution  $x = \frac{1}{x'}$ , it becomes

$$\frac{d^2y'}{dx'^2} + \frac{dy'}{dx'} \left[ \frac{2}{x'} - \frac{q'_1}{x'^2} \right] + \frac{q'_2}{x'^4} y' = 0. \quad (7)$$

And observing that for  $x' = 0$ ,

$$\frac{q'_1}{x'^2} \cdot x' = \phi(0) - 2\delta_1 - 2\delta_2,$$

$$\frac{q'_2}{x'^4} \cdot x'^2 = \psi(0) + (\delta_1 + \delta_2)^2 + (\delta_1 + \delta_2)(1 - \phi(0)).$$

The indicial equation corresponding to the point  $x' = 0$  is

$$r^2 + r[-\phi(0) + 2\delta_1 + 2\delta_2 + 1] + \psi(0) + (\delta_1 + \delta_2)^2 + (\delta_1 + \delta_2)[1 - \phi(0)] = 0. \quad (8)$$

Calling the roots of equation (8)  $\sigma$  and  $\sigma'$ , we find

$$\sigma + \sigma' = \phi(0) - 1 - 2\delta_1 - 2\delta_2,$$

$$\sigma\sigma' = \psi(0) + (\delta_1 + \delta_2)(1 - \phi(0)) - (\delta_1 + \delta_2)^2.$$

Again, if the roots of equation (6) are  $\alpha$  and  $\alpha'$ ,

$$\begin{aligned}\alpha + \alpha' &= \phi(0) - 1, \\ \alpha\alpha' &= \psi(0).\end{aligned}$$

Hence

$$\begin{aligned}\sigma + \sigma' &= (\alpha - \delta_1 - \delta_2) + (\alpha' - \delta_1 - \delta_2), \\ \sigma\sigma' &= (\alpha - \delta_1 - \delta_2)(\alpha' - \delta_1 - \delta_2).\end{aligned}$$

Whence

$$\begin{aligned}\sigma &= \alpha - \delta_1 - \delta_2, \\ \sigma' &= \alpha' - \delta_1 - \delta_2.\end{aligned}$$

Therefore, when the singular points of the integral are  $a, b, \infty$ , to multiply the integral by the factor  $(x-a)^{\delta_1}(x-b)^{\delta_2}$  diminishes the roots of the indicial equation for the point  $\infty$  by  $\delta_1 + \delta_2$  and leaves their difference unchanged.

SECTION 5.—*Determination of the Coefficients of the Differential Equation Satisfied*

$$\text{by } P \left\{ \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array} \right\} x.$$

From the general theory of linear differential equations, since the integral  $P$  is regular and has three singular points, the coefficients of the equation must conform to the following conditions:

1. The coefficient of  $\frac{dy}{dx}$  will be a rational fraction whose numerator cannot be of a degree exceeding  $3-1$ , three being the number of poles, and the denominator is  $(x-a)(x-b)(x-c)$ .
2. The numerator of the coefficient of  $y$  cannot be of higher degree than  $2(3-1)=4$ , and the denominator is  $(x-a)^2(x-b)^2(x-c)^2$ .
3. The constants of the coefficients must be so related that after effecting the transformation  $x = \frac{1}{x'}$ , the point  $x'=0$  shall not be a singular point for the equation, otherwise  $x=\infty$  would be a singular point for the original equation, contrary to hypothesis.

In conformity with these conditions, we may assume the equation to be,  $P, A, \dots, P_1, A_1, \dots$ , being constants,

$$\frac{d^2y}{dx^2} + \frac{P + Ax + Bx^2}{(x-a)(x-b)(x-c)} \frac{dy}{dx} + \frac{P_1 + A_1x + B_1x^2 + C_1x^3 + D_1x^4}{(x-a)^2(x-b)^2(x-c)^2} y = 0. \quad (1)$$

Making the transformation  $x = \frac{1}{x'}$ , this becomes

$$\frac{d^2 y}{dx'^2} + \frac{dy}{dx'} \left[ \frac{2}{x'} - \frac{1}{x'^2} \cdot \frac{x'^2 \left( P + \frac{A}{x'} + \frac{B}{x'^2} \right)}{(1 - ax')(1 - bx')(1 - cx')} \right] \\ + \frac{y}{x'^4} \left[ \frac{x'^2 \left( P_1 + \frac{A_1}{x'_1} + \frac{B_1}{x'^2} + \frac{C_1}{x'^3} + \frac{D_1}{x'^4} \right)}{(1 - ax')^2(1 - bx')^2(1 - cx')^2(1 - dx')^2} \right] = 0. \quad (2)$$

In order that neither coefficient become infinite for  $x' = 0$ , we must have

$$2 - B = 0, \\ C_1 = D_1 = 0;$$

hence equation (1) reduces to

$$\frac{d^2 y}{dx^2} + \frac{P + Ax + 2x^2}{(x - a)(x - b)(x - c)} \frac{dy}{dx} + \frac{P_1 + A_1 x + B_1 x^2}{(x - a)^2(x - b)^2(x - c)^2} y = 0, \quad (3)$$

which may be written

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} \left[ \frac{L}{x - a} + \frac{M}{x - b} + \frac{N}{x - c} \right] \\ + \frac{y}{(x - a)(x - b)(x - c)} \left[ \frac{L_1}{x - a} + \frac{M_1}{x - b} + \frac{N_1}{x - c} \right] = 0, \quad (4)$$

where  $L, \dots, L_1, \dots$  are new constants.

The indicial equations at the points  $a, b, c$  are respectively

$$r^2 + r(L - 1) + \frac{L_1}{(a - b)(a - c)} = 0, \\ r^2 + r(M - 1) + \frac{M_1}{(b - a)(b - c)} = 0, \\ r^2 + r(N - 1) + \frac{N_1}{(c - a)(c - b)} = 0,$$

and by hypothesis, their roots are  $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$  respectively. Therefore

$$L = 1 - \alpha - \alpha'; \quad L_1 = \alpha\alpha'(a - b)(a - c); \\ M = 1 - \beta - \beta'; \quad M_1 = \beta\beta'(b - a)(b - c); \\ N = 1 - \gamma - \gamma'; \quad N_1 = \gamma\gamma'(c - a)(c - b).$$

If  $a = 1, c = 0$ , these values become

$$L = 1 - \alpha - \alpha'; \quad L_1 = \alpha\alpha'(1 - b), \\ M = 1 - \beta - \beta'; \quad M_1 = \beta\beta'(b - 1)b, \\ N = 1 - \gamma - \gamma'; \quad N_1 = \gamma\gamma' \cdot b,$$

and equation (4) takes the form

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} \left[ \frac{L}{x-1} + \frac{M}{x-b} + \frac{N}{x} \right] + \frac{y}{x(x-1)(x-b)} \left[ \frac{L_1}{x-1} + \frac{M_1}{x-b} + \frac{N_1}{x} \right] = 0. \quad (5)$$

Let us now transform (5) by the substitution

$$x' = \frac{x(1-b)}{x-b}$$

such that when

$$\begin{aligned} x=1, & \quad x'=1, \\ x=0, & \quad x'=0, \\ x=b, & \quad x'=\infty. \end{aligned}$$

From this

$$\begin{aligned} x &= \frac{bx'}{x'-(1-b)}; \quad x-b = \frac{b(1-b)}{x'-(1-b)}; \quad x-1 = \frac{(b-1)(x'-1)}{x'-(1-b)}, \\ \frac{dy}{dx} &= -\frac{dy}{dx'} \cdot \frac{(x'-(1-b))^2}{b(1-b)}, \\ \frac{d^2y}{dx^2} &= \frac{d^2y}{dx'^2} \cdot \frac{(x'-(1-b))^4}{b^2(1-b)^2} + 2 \frac{dy}{dx'} \cdot \frac{(x'-(1-b))^3}{b^2(1-b)^2}. \end{aligned}$$

Introducing these values, equation (5) becomes

$$\begin{aligned} \frac{[x'-(1-b)]^4}{b^2(1-b)^2} \frac{d^2y}{dx'^2} + \frac{dy}{dx'} \left[ 2 \frac{[x'-(1-b)]^3}{b^2(1-b)^2} \right. \\ \left. - \frac{[x'-(1-b)]^2}{b(1-b)} \left( L \frac{x'-(1-b)}{(b-1)(x'-1)} + M \frac{x'-(1-b)}{b(1-b)} + N \frac{x'-(1-b)}{bx'} \right) \right] \\ - y \frac{[x'-(1-b)]^3}{b^2(1-b)^2 x'(x'-1)} \left[ L_1 \frac{x'-(1-b)}{(b-1)(x'-1)} \right. \\ \left. + M_1 \frac{x'-(1-b)}{b(1-b)} + N_1 \frac{x'-(1-b)}{bx'} \right] = 0. \end{aligned}$$

Noting that

$$\begin{aligned} L \frac{x'-(1-b)}{(b-1)(x'-1)} + M \frac{x'-(1-b)}{b(1-b)} + N \frac{x'-(1-b)}{bx'} \\ = \frac{x'-(1-b)}{b(1-b)(x'-1)x'} [-Lbx' + Mx'(x'-1) + N(1-b)(x'-1)] \\ = \frac{x'-(1-b)}{b(1-b)x'(x'-1)} [Mx'^2 + x'((1-b)(N+L) - L - M) + N(1-b)], \end{aligned}$$

and subtracting the quantity in the bracket from  $2x'(x' - 1)$ , which has now the same coefficient, we get

$$(2 - M)x' + x'[L + M - (1 - b)(N + L) - 2] + N(1 - b). \quad (6)$$

Now

$$L + M - 2 = -N$$

and

$$N + L = 2 - L,$$

remembering that  $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$ .

Hence the expression (6) breaks up into the factors

$$[x' - (1 - b)][(2 - M)x' - N].$$

Again, noting that

$$\frac{L_1}{b - 1} = -\alpha\alpha'; \quad \frac{M_1}{b(1 - b)} = -\beta\beta'; \quad \frac{N_1}{b} = \gamma\gamma',$$

and observing that the factor  $\frac{[x' - (1 - b)]^4}{b^3(1 - b)^3}$  is now common to every term of the equation, we obtain

$$\frac{d^2y}{dx'^2} + \frac{dy}{dx'} \frac{x'(2 - M) - N}{x'(x' - 1)} - \frac{y}{x'(x' - 1)} \left[ \frac{-\alpha\alpha'}{x' - 1} - \beta\beta' + \frac{\gamma\gamma'}{x'} \right] = 0, \quad (7)$$

or finally,

$$\frac{d^2y}{dx'^2} + \frac{dy}{dx'} \frac{x'(1 + \beta + \beta') - 1 + \gamma + \gamma'}{x'(x' - 1)} - \frac{y}{x'(x' - 1)} \left[ \frac{\alpha\alpha'}{1 - x'} - \beta\beta' + \frac{\gamma\gamma'}{x'} \right] = 0. \quad (8)$$

If we had chosen  $a = 0$ ,  $b = \infty$ ,  $c = 1$ , the equation would have been

$$\frac{d^2y}{dx'^2} + \frac{dy}{dx'} \frac{\alpha + \alpha' - 1 + x'(1 + \beta + \beta')}{x'(x' - 1)} + y \frac{\alpha\alpha' + x'(\gamma\gamma' - \alpha\alpha' - \beta\beta') + \beta\beta'x'^2}{x'^3(1 - x')^2} = 0, \quad (9)$$

which is Papperitz's form.

SECTION 6.— $\alpha = 0$ ,  $\alpha' = \frac{1}{2}$ ;  $\beta + \beta' + \gamma + \gamma' = \frac{1}{2}$ .

To this case we may reduce that of one difference  $= \frac{1}{2}$ . Substituting these values in the differential equation, it becomes

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} \frac{x(1 + \beta + \beta') - \frac{1}{2}}{x(x - 1)} + y \frac{\beta\beta'x^2 + x(\gamma\gamma' - \beta\beta')}{x^2(1 - x)^2} = 0. \quad (1)$$

From this, if  $x = z^2$ , we obtain the equation

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} \frac{z(1 + 2\beta + 2\beta')}{z(z^2 - 1)} + 4y \frac{\beta\beta'z^2 + \gamma\gamma' - \beta\beta'}{(1 - z^2)^2} = 0. \quad (2)$$

Effecting the transformation  $z = \frac{1}{z'}$  in (2), we obtain

$$\frac{d^2y}{dz'^2} + \frac{dy}{dz'} \left[ \frac{2}{z'} - \frac{1 + 2\beta + 2\beta'}{z'(1 - z'^2)} \right] + 4y \frac{\beta\beta'z'^2 + (\gamma\gamma' - \beta\beta')z'}{z'^4(z'^2 - 1)^2} = 0, \quad (3)$$

which shows that  $z' = 0$  is a pole of multiplicity, 1 for the first and 2 for the second coefficient. The poles of (2) are consequently  $+1$ ,  $-1$ , and  $\infty$ , and the corresponding indicial equations are respectively

$$r^2 + r(\beta + \beta' - \frac{1}{2}) + \gamma\gamma' = 0, \quad (4)$$

$$r^2 + r(\beta + \beta' - \frac{1}{2}) + \gamma\gamma' = 0, \quad (5)$$

$$r^2 + r(-2\beta - 2\beta') + 4\beta\beta' = 0. \quad (6)$$

But, remembering that

$$\beta + \beta' + \gamma + \gamma' = \frac{1}{2},$$

these equations reduce to

$$r^2 - r(\gamma + \gamma') + \gamma\gamma' = 0, \quad (7)$$

$$r^2 - r(\gamma + \gamma') + \gamma\gamma' = 0, \quad (8)$$

$$r^2 - 2r(\beta + \beta') + 4\beta\beta' = 0, \quad (9)$$

of which the roots are respectively

$$\gamma, \gamma'; \gamma, \gamma'; 2\beta, 2\beta';$$

that is, by the transformation  $x = z^2$  the equation satisfied by  $P\left(\begin{smallmatrix} 0 & \beta & \gamma \\ \frac{1}{2} & \beta' & \gamma' \end{smallmatrix} x\right)$  becomes an equation satisfied by  $P\left(\begin{smallmatrix} \gamma & 2\beta & \gamma \\ \gamma' & 2\beta' & \gamma' \end{smallmatrix} z\right)$ , which is a result obtained in Part I, Section 5.

In equation (10), Section 5, let us make  $\alpha' = \beta' = \gamma' = 0$  and  $\alpha = \beta = \nu$ , while  $\gamma = 1$ , we thus obtain

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1 - \nu}{x} = 0. \quad (10)$$

The general integral of this equation is

$$cx^\nu + c',$$

where  $c$  and  $c'$  are constants. Hence, we conclude with Riemann that

$$P\left(\begin{smallmatrix} 0 & 0 & 0 \\ \nu & -\nu & 1 \end{smallmatrix} x\right) = cx^\nu + c'.$$

SECTION 7.—*Spherical Harmonics expressed as P-functions.*

In equation (2), Section 6, the quantities  $\beta, \beta', \gamma, \gamma'$  are connected by the relation

$$\beta + \beta' + \gamma + \gamma' = \frac{1}{2}. \quad (1)$$

If we make  $\gamma' = 0$ , (1) becomes  $\beta + \beta' + \gamma = \frac{1}{2}$ , a relation which is identically satisfied by the values

$$\gamma = 0, \quad \beta = \frac{n+1}{2}, \quad \beta' = -\frac{n}{2}.$$

These values reduce equation (2) of Section 6 to

$$\frac{d^2 y}{dz^2} + \frac{dy}{dz} \frac{2z}{z^2-1} - \frac{n(n+1)}{z^2-1} y = 0,$$

which is the differential equation satisfied by the zonal spherical harmonics, if we apply that name to the function  $\frac{d^n}{dz^n} (z^2-1)^n$  (Jordan, Cours d'Analyse, Vol. 1, p. 51). Hence the *P*-function

$$P \left\{ \begin{matrix} 1 & \infty & -1 \\ 0 & n+1 & 0 \\ 0 & -n & 0 \end{matrix} \middle| z \right\}$$

represents the zonal harmonic of order  $n$  (see Ferrers, Spherical Harmonics, p. 12).

SECTION 8.—*The Toroidal Functions expressed as P-functions.*

In equation (2) of Section 6 let us make

$$\begin{aligned} \gamma + \gamma' &= 0, \\ 4\gamma\gamma' &= -m^2. \end{aligned}$$

Then shall

$$\gamma = -\gamma' = \pm \frac{m}{2}.$$

And from the relation

$$\beta + \beta' + \gamma + \gamma' = \frac{1}{2}$$

we find

$$\beta + \beta' = \frac{1}{2}.$$

Assuming also

$$\beta - \beta' = n,$$

we get

$$4\beta\beta' = -n^2 + \frac{1}{4},$$

and also

$$1 + 2\beta + 2\beta' = 2.$$

Substituting these values in equation (2), it becomes

$$\frac{d^2y}{dz^2} + \frac{2}{z^2-1} \frac{dy}{dz} + \frac{(n^2-\frac{1}{4})(1-z^2)-m^2}{(1-z^2)^2} y = 0, \quad (1)$$

which is the differential equation satisfied by the toroidal functions (Basset, *Hydrodynamics*, Vol. 2, p. 22).

Observing that  $\beta = \frac{n}{2} + \frac{1}{4}$ ,  $\beta' = \frac{1}{4} - \frac{n}{2}$ , we see that the *P*-function

$$P \left\{ \begin{matrix} 1 & \infty & -1 \\ \frac{m}{2} & n + \frac{1}{2} & \frac{m}{2} \\ -\frac{m}{2} & \frac{1}{2} - n & -\frac{m}{2} \end{matrix} \middle| z \right\}$$

represents the toroidal functions.

In equation (1), making  $m = 0$ , which corresponds to  $\gamma = \gamma' = 0$ , we get the equation for zonal toroidal functions,

$$\frac{d^2y}{dz^2} + \frac{2}{z^2-1} \frac{dy}{dz} + \frac{n^2-\frac{1}{4}}{1-z^2} y = 0, \quad (2)$$

which has for its integral the *P*-function

$$P \left\{ \begin{matrix} 1 & \infty & -1 \\ 0 & n + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - n & 0 \end{matrix} \middle| z \right\}.$$

#### SECTION 9.—*Bessel's Equation.*

In equation (10) of Section 6 let us make the substitution

$$x = \varepsilon z, \quad (1)$$

then shall

$$\frac{dy}{dx} = \frac{1}{\varepsilon} \frac{dy}{dz},$$

$$\frac{d^2y}{dx^2} = \frac{1}{\varepsilon^2} \frac{d^2y}{dz^2},$$

and the equation becomes

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} \cdot \frac{\varepsilon z(1+\beta+\beta')+\alpha+\alpha'-1}{z(\varepsilon z-1)} + y \frac{\alpha\alpha'+\varepsilon z(\gamma\gamma'-\alpha\alpha'-\beta\beta')+\beta\beta'\varepsilon^2z^2}{z^2(\varepsilon z-1)^2} = 0. \quad (2)$$



Let, now,  $\varepsilon$  tend toward zero and  $\beta, \beta'$  toward infinity in such a manner that the product  $\varepsilon^2 \beta \beta'$  shall be constantly equal to 1; that is,

$$\beta \beta' = \frac{1}{\varepsilon^2},$$

and also let

$$\alpha \alpha' = -m^2, \text{ a constant,}$$

$$\gamma \gamma' = \beta \beta',$$

$$\alpha + \alpha' - 1 = -1,$$

$$\beta + \beta' = 0.$$

These equations, together with

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1,$$

enable us to determine the six exponents; thus we find

$$\alpha = \pm m, \alpha' = \mp m,$$

$$\beta = \pm \frac{1}{\varepsilon} \sqrt{-1}; \beta' = \mp \frac{1}{\varepsilon} \sqrt{-1},$$

$$\gamma = \frac{1}{2} \left( 1 \pm \frac{1}{\varepsilon} \sqrt{\varepsilon^2 - 4} \right); \gamma' = \frac{1}{2} \left( 1 \mp \frac{1}{\varepsilon} \sqrt{\varepsilon^2 - 4} \right),$$

and equation (2) becomes

$$\frac{d^2 y}{dz^2} + \frac{1}{z} \frac{dy}{dz} + y \left( 1 - \frac{m^2}{z^2} \right) = 0,$$

which is Bessel's equation.

We conclude, therefore, that the limiting value of

$$P \left\{ \begin{matrix} 0 & \infty & 1 \\ \pm m & \pm \frac{1}{\varepsilon} \sqrt{-1} & \frac{1}{2} \left( 1 \pm \frac{1}{\varepsilon} \sqrt{\varepsilon^2 - 4} \right) \\ \mp m & \mp \frac{1}{\varepsilon} \sqrt{-1} & \frac{1}{2} \left( 1 \mp \frac{1}{\varepsilon} \sqrt{\varepsilon^2 - 4} \right) \end{matrix} \right\}$$

when  $\varepsilon = 0$  is the Bessel's function  $J_m(z)$ .

#### SECTION 10.—The *P*-function expressed as a Hypergeometric Series.

Equation (10) of Section 6 is

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} \frac{x(1 + \beta + \beta') + \alpha + \alpha' - 1}{x(x-1)} + y \frac{\alpha \alpha' + x'(\gamma \gamma' - \alpha \alpha' - \beta \beta') + \beta \beta' x^2}{x^2(1-x)^2} = 0. \quad (1)$$

In the region of the point  $x = 0$  the indicial equation is

$$r^2 - r(\alpha + \alpha') + \alpha\alpha' = 0, \quad (2)$$

of which the roots are  $\alpha$  and  $\alpha'$ . Knowing that the integral does not contain logarithms, we may assert that in the region of the point 0, equation (1) will be satisfied by a convergent series of the form

$$x^\alpha (c_0 + c_1x + c_2x^2 + \dots + c_\mu x^\mu + \dots).$$

Before substituting this series in equation (1) it will be convenient to transform the equation as follows. Put  $\alpha - \alpha' = \lambda$ ,  $\beta - \beta' = \mu$ ,  $\gamma - \gamma' = \nu$ , and keeping  $\lambda, \mu, \nu$  constant, let us determine  $\alpha, \beta, \gamma$  so that the following conditions may be satisfied:

$$\alpha\alpha' = 0, \quad \gamma\gamma' = 0.$$

Consistently with these conditions we make

$$\begin{aligned} \alpha' &= 0, \quad \alpha = \lambda, \\ \gamma' &= 0, \quad \gamma = \mu, \end{aligned}$$

and accordingly

$$\begin{aligned} \beta + \beta' &= 1 - \lambda - \nu, \\ \beta - \beta' &= \mu. \end{aligned}$$

Hence

$$\left. \begin{aligned} 2\beta &= 1 - \lambda - \nu + \mu, \\ 2\beta' &= 1 - \lambda - \nu - \mu, \\ \beta\beta' &= \frac{(1 - \lambda - \nu)^2 - \mu^2}{4}. \end{aligned} \right\} \quad (3)$$

With these values, equation (1) becomes

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} \frac{\lambda - 1 + (2 - \lambda - \nu)x}{x(x-1)} + y \frac{(1 - \lambda - \nu)^2 - \mu^2}{4x(x-1)} = 0, \quad (4)$$

since the numerator of the last coefficient becomes divisible by  $x(x-1)$ .

For the point  $x = 0$ , the indicial equation is now

$$r^2 - r\lambda = 0, \quad (5)$$

of which the roots are 0 and  $\lambda$ . Hence the integrals in the region of the point  $x = 0$  are of the form

$$x^\lambda (c_0 + c_1x + c_2x^2 + \dots)$$

and

$$a_0 + a_1x + a_2x^2 + \dots$$

Substituting the second series, we find, by equating to 0 the coefficient of  $x^r$ ,

$$C_{r+1} = -C_r \frac{x^2 + x(1 - \lambda - \nu) + \frac{1}{4}(1 - \lambda - \nu + \mu)(1 - \lambda - \nu - \mu)}{(x + 1)(\lambda - 1 - x)}.$$

Making

$$1 - \lambda - \mu + \nu = 2a,$$

$$1 - \lambda - \mu - \nu = 2b,$$

$$1 - \lambda = c,$$

this relation becomes

$$C_{r+1} = C_r \frac{(a + x)(b + x)}{(x + 1)(c + x)},$$

and the integral is, by making  $C_0 = 1$ ,

$$1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot (c+1)}x^2 + \dots = F(a, b, c, x),$$

where  $F(a, b, c, x)$  denotes the hypergeometric series.

Likewise, by substituting the first series, we find

$$C_{r+1} = C_r \frac{\left(x + \frac{1 + \lambda - \nu - \mu}{2}\right)\left(x + \frac{1 + \lambda - \nu + \mu}{2}\right)}{(1 + x)(x + 1 + \lambda)},$$

wherein, if

$$\frac{1 + \lambda - \nu + \mu}{2} = a',$$

$$\frac{1 + \lambda - \nu - \mu}{2} = b', \quad 1 + \lambda = c',$$

we obtain the relation

$$C_{r+1} = C_r \frac{(a' + x)(b' + x)}{(1 + x)(c' + x)}.$$

Hence the second integral in the region of the point  $x = 0$  is

$$x^\lambda F(a', b', c', x),$$

we conclude finally that in the region of the point 0

$$P(\lambda, \mu, \nu, x) = C_1 F\left(\frac{1 - \lambda - \nu + \mu}{2}, \frac{1 - \lambda - \nu - \mu}{2}, 1 - \lambda, x\right) \\ + C_2 x^\lambda F_1\left(\frac{1 + \lambda - \nu + \mu}{2}, \frac{1 + \lambda - \nu - \mu}{2}, 1 + \lambda, x\right),$$

where  $C_1$  and  $C_2$  are arbitrary constants.

## AUTOBIOGRAPHY.

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I, Charles Hiram Chapman, was born October 28, 1859, in Columbia County, Wisconsin. When I was eight years old my parents removed to Viroqua, Wisconsin, where they have since lived.

I attended the public schools in Viroqua until I was fourteen, and then, after teaching one term of district school, I entered Galesville University, Trempealeau County, Wisconsin, where I stayed a year, studying principally Latin and Greek.

From that time until the fall of 1886 I taught school in various parts of Wisconsin, Nebraska and Minnesota, occupying my leisure hours by the study of mathematics.

In the fall of 1886 I entered the Normal School at Oshkosh, Wisconsin, and graduated thence the following spring.

At the beginning of the session of 1887-88 I entered Johns Hopkins University and took the degree of Bachelor of Arts. Since then I have pursued my studies in Mathematics, Astronomy and Physics in that University.



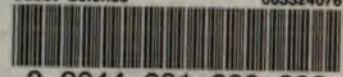
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